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Impact response of a viscoelastic beam considering the changes of its microstructure in the contact domain

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Abstract The problem on low-velocity impact of a long thin elastic rod with a flat end upon an infinite viscoelastic Timoshenko-type beam, the dynamic behaviour of which is described by a set of equations taking the rotary inertia, transverse shear deformation and extension of the beam's middle surface into account, is considered. The viscoelastic features of the beam are governed by the standard linear solid model with derivatives of integer order. At the moment of impact, shock waves (surfaces of strong discontinuity) are generated both in the impactor and target, the influence of which on the contact domain is considered via the theory of discontinuities. The contact zone moves like a rigid whole under the action of the contact force and longitudinal and transverse forces applied to the boundary of the contact region, which are obtained on the basis of one-term ray expansions. During the impact process, decrosslinking within the domain of the contact of the beam with the rod occurs, resulting in more free displacements of molecules with respect to each other, and finally in the decrease of the beam material viscosity in the contact zone. This circumstance allows one to describe the behaviour of the beam material within the contact domain by the standard linear solid model involving fractional derivatives, since variation in the fractional parameter (the order of the fractional derivative) enables one to control the viscosity of the beam material. The contact force has been determined analytically via the Laplace transform technique.

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1 Introduction

The problems connected with the analysis of the shock interaction of thin bodies (rods, beams, plates, and shells) with other bodies have widespread application in various fields of science and technology. The physical phenomena involved in the impact event include structural responses, contact effects and wave propagation. These problems are topical not only from the point of view of fundamental research in applied mechanics, but also with respect to their applications. Because these problems belong to the problems of dynamic contact interaction, their solution is connected with severe mathematical and calculation difficulties. To overcome this impediment, a rich variety of approaches and methods have been suggested, and the overview of current results in the field can be found in recent state-of-the-art articles (Abrate 2001; Rossikhin and Shitikova 2007b, 2010, 2013).

All impactors according to their geometry could be divided into two types, namely: with a flat end and with a rounded end, resulting in different approaches for solving contact/impact problems (Rossikhin and Shitikova 2007b), since the contact domain remains unchanged in time in the first case, and it is a time-dependent function in the second case. The impactors with rounded ends are considered in the majority of papers devoted to the impact interaction of solids, and the Hertz contact law or its modifications is used for defining the contact force.

But in many practical applications plane-ended indentors and impactors are used, and there is a need to analyse the contact force arising in such problems especially when viscoelastic features of the impactors and/or targets should be taken into account. Thus, Argatov et al. (2013) considered a viscoelastic layer bonded to a rigid substrate indented by a flat-ended cylindrical indenter. Argatov (2013) studied an axisymmetric contact problem for a thin biphasic layer indented without friction by a rigid impermeable cylindrical indenter. Perturbation analysis of the impact process according to the standard viscoelastic solid model was performed in Argatov (2013), wherein asymptotic solutions are obtained for the drop weight impact test. Sinusoidally-driven flat-ended indentation of time-dependent materials was investigated in Argatov (2012), and simple asymptotic models for low and high rate loading were suggested.

An exact analytical solution for the problem of a semi-infinite elastic rod struck by a rigid mass through a linear Kelvin–Voigt element was presented in Argatov and Jokinen (2013). Axial impact between a cylindrical striker of finite length and a long cylindrical bar, both of linearly viscoelastic materials, was considered by Bussac et al. (2008). A computational model capable of handling viscoelastic contact-impact problems was proposed in Assie et al. (2010), and one, two and three-dimensional finite element models were illustrated by longitudinal impact of two viscoelastic bars, by the analysis of the impact of two sheets and two blocks, the viscoelastic features of which are described by the standard linear solid model,

In recent decades fractional calculus (integral and differential operators of noninteger order), which has a long history (Valério et al. 2014), has been the object of ever increasing interest in many branches of natural science, and of engineering interest as well. Thus, Rossikhin and Shitikova (2010) have reviewed the application of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids, among them, the problems of dynamic contact interaction. Two approaches have been discussed for studying the impact response of fractionally damped systems subjected to falling impactors

(Rossikhin and Shitikova 2013). The first one is based on the assumption that viscoelastic properties of the target manifest themselves only in the contact domain, while the other part of the target remains elastic one. This approach results in defining the contact force and the local penetration of target by an impactor from the set of linear fractional differential equations. The second approach is the immediate generalization of the Timoshenko approach utilising the viscoelastic analog of Hertz's contact law by substituting elastic constants by the corresponding viscoelastic operators. This approach results in the nonlinear functional equation for determining the contact force or the impactor's relative displacement.

In the present paper, dynamic response of a viscoelastic beam impacted by a long plane-ended elastic rod is studied using two approaches. The first one is the development of the previous analysis carried out in Rossikhin and Shitikova (1996, 2007b) for an elastic isotropic Timoshenko beam subjected to the impact by an elastic long rod by considering a hereditarily elastic Timoshenko-like beam impacted by an elastic prismatic long rod of a rectangular cross-section.

The second approach is based on the dynamic theory of the behaviour of thin bodies, which differs from the Timoshenko theory and which was recently proposed in Rossikhin and Shitikova (2007a, 2008, 2011). This theory is based on three-dimensional equations of the material from which a thin body is made of, on the theory of discontinuities based on the conditions of compatibility suggested by the authors, and on the classical assumptions for thin bodies, namely, the hypothesis about plane sections or rigid profile, non-press of layers, and so on. Within the framework of this theory, a set of recurrent equations in discontinuities of arbitrary order in time for desired values is deduced, which allows one to construct with a help of the ray series the solution of boundary-value problems dealing with the transient dynamic loads on thin bodies, resulting in propagation of transient waves (surfaces of discontinuities) in the thin bodies. This theory is free from additional constants, like shear coefficients in the Timoshenko theory which depend on the geometry of thin body's cross-section and are not determined experimentally, and is based only on the material's constants.

In both approaches used in the given paper, the viscoelastic properties of the beam are described by the standard linear solid model with integer time-derivatives. During the impact process, decrosslinking within the domain of the contact of the beam with the rod occurs, resulting in more free displacements of molecules with respect to each other, and finally in the decrease of the beam material viscosity in the contact zone. This circumstance allows one to describe the behaviour of the beam material within the contact domain by the standard linear solid model involving fractional derivatives, since variation in the fractional parameter (the order of the fractional derivative) enables one to control the viscosity of the beam material.

2 Problem formulation and governing equations

Let a long prismatic elastic rod of a rectangular cross-section with the dimensions $2\tau_{\rm im}$ and a move along the y-normal with the velocity V_0 towards a viscoelastic homogeneous isotropic rectangular Timoshenko beam of infinite extent (this assumption is introduced due to the short duration of contact interaction in order to ignore reflected waves) with width a and thickness h, in so doing the normal y is erected at the middle of the beam (Fig. 1(a)). The viscoelastic features of the beam are described by the standard linear solid model (Fig. 1(b)).

The dynamic behaviour of such a beam is described by the following set of equations:

$$\frac{\partial N}{\partial z} = \rho F \dot{v}_z, \tag{1}$$

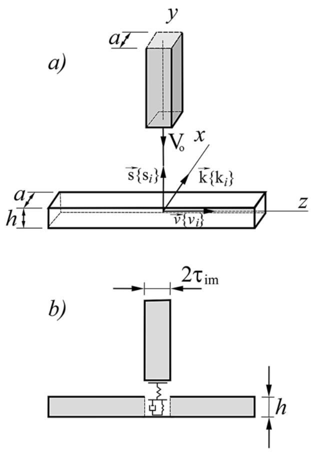

Fig. 1 Scheme of the shock interaction of a viscoelastic beam and the plain-ended impactor

$$\frac{\partial Q}{\partial z} = \rho F \dot{v}_y, \tag{2}$$

$$\frac{\partial M}{\partial z} - Q = -\rho I \dot{\Psi}, \tag{3}$$

$$N = F E_\infty \big[1 - \nu_\varepsilon \ni_1^* (\tau_\varepsilon)\big] \frac{\partial u_z}{\partial z}, \tag{4}$$

$$Q = K F \mu_\infty \big[1 - n \ni_1^* (t_\varepsilon)\big] \left(\frac{\partial u_y}{\partial z} - \psi\right), \tag{5}$$

$$M = -I E_\infty \big[1 - \nu_\varepsilon \ni_1^* (\tau_\varepsilon)\big] \frac{\partial \psi}{\partial z}, \tag{6}$$

where M, Q, and N are the bending moment, the shear and longitudinal forces, respectively, u_z and u_y are longitudinal and transverse displacements, respectively, ψ is the angle of rotation of the cross-section around the z-axis, $v_z = \dot{u}_z$, $v_y = \dot{u}_y$, $\Psi = \dot{\psi}$, F and I are the cross-sectional area and the moment of inertia with respect to the x-axis, respectively, ρ is the density, K is the shear coefficient dependent on beam's geometrical dimensions and the form of its cross-section, and an overdot denotes the time derivative.

In Eqs. (4) and (6), the operator corresponding to the Young modulus has the form

$$\widetilde{E} = E_\infty \big[1 - \nu_\varepsilon \ni_1^* (\tau_\varepsilon)\big], \tag{7}$$

$$\nu_\varepsilon = \frac{E_\infty - E_0}{E_\infty} = \frac{\Delta E}{E_\infty}, \tag{8}$$

$$\ni_1^* (\tau_\varepsilon) Z(t) = \frac{1}{\tau_\varepsilon} \int_0^t e^{-(t-t')/\tau_\varepsilon} Z\big(t'\big)\, \mathrm{d}t', \tag{9}$$

where $Z(t)$ is a desired function, E_∞ and E_0 are the non-relaxed (instantaneous modulus of elasticity, or the glassy modulus) and relaxed elastic (prolonged modulus of elasticity, or the rubbery modulus) moduli which are connected with the relaxation time τ_ε and retardation time τ_σ by the following relationship:

$$\frac{\tau_\varepsilon}{\tau_\sigma} = \frac{E_0}{E_\infty}. \tag{10}$$

In Eq. (5), the operator corresponding to the shear modulus has the form

$$\widetilde{\mu} = \mu_\infty\big[1 - n \ni_1^* (t_\varepsilon)\big], \tag{11}$$

where μ_∞ is the non-relaxed magnitude of the shear modulus, and n and t_ε are for now unknown constants.

In order to find the inverse operator $\widetilde{E}^{-1}$, as well as other operators important for further treatment, it is necessary to obtain formulas governing the product of operators $\ni_\gamma^* (\tau_i)$ $(i = \varepsilon, \sigma)$.

For this purpose let us differentiate (6) with respect to t. As a result we obtain

$$\dot{M} = -I E_\infty\left[\frac{\partial \Psi}{\partial z} - \frac{\nu_\varepsilon}{\tau_\varepsilon}\frac{\partial \psi}{\partial z} + \frac{\nu_\varepsilon}{\tau_\varepsilon^2}\int_0^t e^{-(t-t')/\tau_\varepsilon}\frac{\partial \psi(t')}{\partial z}\,\mathrm{d}t'\right]. \tag{12}$$

Eliminating $\int_0^t e^{-(t-t')/\tau_\varepsilon}\frac{\partial\psi(t')}{\partial z}\,\mathrm{d}t'$ from (6) and (12) yields

$$M + \tau_\varepsilon \dot{M} = -I E_\infty\left[(1-\nu_\varepsilon)\frac{\partial \psi}{\partial z} + \tau_\varepsilon\frac{\partial \Psi}{\partial z}\right], \tag{13}$$

or accounting for (8) and (10),

$$M + \tau_\varepsilon \dot{M} = -I E_0\left(\frac{\partial \psi}{\partial z} + \tau_\sigma\frac{\partial \Psi}{\partial z}\right). \tag{14}$$

Let us rewrite formula (14) in the form

$$M = -I E_\infty\frac{E_0 E_\infty^{-1} + \tau_\varepsilon\,\mathrm{d}/\mathrm{d}t}{1 + \tau_\varepsilon\,\mathrm{d}/\mathrm{d}t}\frac{\partial \psi}{\partial z}. \tag{15}$$

Let us add and subtract a unit in the numerator of (15) and then divide the numerator term-wise by the denominator $1 + \tau_\varepsilon\,\mathrm{d}/\mathrm{d}t$. As a result we obtain

$$M = -I E_\infty\left(1 - \nu_\varepsilon\frac{1}{1 + \tau_\varepsilon\,\mathrm{d}/\mathrm{d}t}\right)\frac{\partial \psi}{\partial z}. \tag{16}$$

Comparing relationships (6) and (16) yields

$$\ni_1^* (\tau_\varepsilon) = \frac{1}{1 + \tau_\varepsilon\,\mathrm{d}/\mathrm{d}t}. \tag{17}$$

On the basis of relationship (17), we can write

$$\ni_1^* (\tau_\varepsilon) \ni_1^* (\tau_\sigma) = \frac{1}{(1 + \tau_\varepsilon\,\mathrm{d}/\mathrm{d}t)(1 + \tau_\sigma\,\mathrm{d}/\mathrm{d}t)} = \frac{A_1}{1 + \tau_\varepsilon\,\mathrm{d}/\mathrm{d}t} + \frac{B_1}{1 + \tau_\sigma\,\mathrm{d}/\mathrm{d}t},$$

where

$$A_1 = \frac{\tau_\varepsilon}{\tau_\varepsilon - \tau_\sigma}, \qquad B_1 = -\frac{\tau_\sigma}{\tau_\varepsilon - \tau_\sigma},$$

whence it follows that

$$\ni_1^* (\tau_\varepsilon) \ni_1^* (\tau_\sigma) = \frac{\tau_\varepsilon \ni_1^* (\tau_\varepsilon) - \tau_\sigma \ni_1^* (\tau_\sigma)}{\tau_\varepsilon - \tau_\sigma}. \tag{18}$$

Relationship (18) represents the theorem of multiplication of the operators $\ni_1^*(\tau)$.

To find the inverse operator $\widetilde{E}^{-1}$, let us represent it in the following form:

$$\widetilde{E}^{-1} = E_\infty^{-1}\big[1 + \nu_\sigma \ni_1^*(\tau_\sigma)\big], \tag{19}$$

where ν_σ and τ_σ are for now unknown constants.

Considering (7), (18), and (19) from the definition of the inverse operator

$$\widetilde{E}\widetilde{E}^{-1} = 1, \tag{20}$$

we find

$$\nu_\sigma\left(1 + \frac{\nu_\varepsilon \tau_\sigma}{\tau_\varepsilon - \tau_\sigma}\right) \ni_1^*(\tau_\sigma) - \nu_\varepsilon\left(1 + \frac{\nu_\sigma \tau_\varepsilon}{\tau_\varepsilon - \tau_\sigma}\right) \ni_1^*(\tau_\varepsilon) = 0. \tag{21}$$

From (21) it follows that

$$1 + \frac{\nu_\varepsilon \tau_\sigma}{\tau_\varepsilon - \tau_\sigma} = 0, \tag{22}$$

$$1 + \frac{\nu_\sigma \tau_\varepsilon}{\tau_\varepsilon - \tau_\sigma} = 0. \tag{23}$$

From (22) and (23) we obtain

$$\tau_\sigma = \tau_\varepsilon \frac{E_\infty}{E_0}, \tag{24}$$

$$\nu_\sigma = \frac{E_\infty - E_0}{E_0} = \frac{J_0 - J_\infty}{J_\infty}, \tag{25}$$

where $J_0 = E_0^{-1}$, and $J_\infty = E_\infty^{-1}$.

Note that formula (24) coincides with formula (10).

Since thin rods are used, as a rule, in creep and relaxation experiments, first of all the operators $\widetilde{E}$ and $\widetilde{E}^{-1}$ are determined from such experiments. As numerous experimental data on viscoelastic materials show (Rabotnov 1966), the operator of volume expansion–contraction for many viscoelastic materials could be considered as time-independent, i.e.,

$$\widetilde{K} = K_\infty, \tag{26}$$

where $K_\infty = \text{const}$.

Since in the present paper we study the transverse impact phenomenon, the shear aftereffect plays the most essential role, while the bulk aftereffect could be ignored. Thus, the above assumption is valid for the problem under consideration.

In other words,

$$\frac{\widetilde{E}}{1 - 2\widetilde{\nu}} = \frac{E_\infty}{1 - 2\nu_\infty} = 3K_\infty, \tag{27}$$

where $\widetilde{\nu}$ is the Poisson operator, and ν_∞ is the non-relaxed Poisson coefficient.

From formula (27) we have

$$\widetilde{\nu} = \nu_\infty + \frac{1}{2}(1 - 2\nu_\infty)\nu_\varepsilon \ni_1^*(\tau_\varepsilon). \tag{28}$$

Knowing the operators $\widetilde{E}$ and $\widetilde{\nu}$, all other operators could be easily calculated. But before defining the operators $\widetilde{\mu}$ and $\widetilde{\lambda}$, first we find the operator inverse to the operator

$$1+\widetilde{\nu}=(1+\nu_\infty)\left[1+\frac{(1-2\nu_\infty)\nu_\varepsilon}{2(1+\nu_\infty)}\ni_1^*(\tau_\varepsilon)\right]. \tag{29}$$

Using expression

$$(1+\widetilde{\nu})^{-1}(1+\widetilde{\nu})=1$$

and considering (18) yields

$$(1+\widetilde{\nu})^{-1}=\frac{1}{1+\nu_\infty}\left[1-D\ni_1^*(t_\sigma)\right], \tag{30}$$

where $t_\sigma=\tau_\varepsilon C^{-1}$,

$$D=\frac{(1-2\nu_\infty)\nu_\varepsilon}{2(1+\nu_\infty)+(1-2\nu_\infty)\nu_\varepsilon}, \qquad C=\frac{2(1+\nu_\infty)+(1-2\nu_\infty)\nu_\varepsilon}{2(1+\nu_\infty)}.$$

Accounting for (7), (18), and (30), we obtain

$$\begin{aligned}\widetilde{\mu}=\frac{\widetilde{E}}{2(1+\widetilde{\nu})}=\mu_\infty\bigg[1-\nu_\varepsilon\left(1-D\frac{\tau_\varepsilon}{\tau_\varepsilon-t_\sigma}\right)\ni_1^*(\tau_\varepsilon)\\ -D\left(1+\nu_\varepsilon\frac{t_\sigma}{\tau_\varepsilon-t_\sigma}\right)\ni_1^*(t_\sigma)\bigg].\end{aligned} \tag{31}$$

Considering that

$$1-D\frac{\tau_\varepsilon}{\tau_\varepsilon-t_\sigma}=0,$$

$$D\left(1+\nu_\varepsilon\frac{t_\sigma}{\tau_\varepsilon-t_\sigma}\right)=\frac{3\nu_\varepsilon}{2(1+\nu_\infty)C},$$

from (31) we have

$$\widetilde{\mu}=\mu_\infty\left[1-n\ni_1^*(t_\sigma)\right], \quad n=\frac{3\nu_\varepsilon}{2(1+\nu_\infty)C}. \tag{32}$$

First we write operator $\widetilde{\lambda}$ in the form

$$\widetilde{\lambda}=\frac{\widetilde{E}\widetilde{\nu}}{(1-2\widetilde{\nu})(1+\widetilde{\nu})}=\frac{1}{3}\frac{\widetilde{E}}{1-2\widetilde{\nu}}-\frac{1}{3}\frac{\widetilde{E}}{1+\widetilde{\nu}},$$

and then, considering formulae (27) and (31), we rewrite it in the form

$$\widetilde{\lambda}=\lambda_\infty\left[1+n_1\ni_1^*(t_\sigma)\right], \quad n_1=\frac{(1-2\nu_\infty)\nu_\varepsilon}{2(1+\nu_\infty)C\nu_\infty}. \tag{33}$$

Note that if a mathematical pendulum is used for experimental measurement of internal friction in a beam, then first of all the operators $\widetilde{\mu}$ and $\widetilde{\mu}^{-1}$ are determined, while the other operators are expressed in terms of them and the constant K_∞ (see Appendix).

The dynamic behaviour of an elastic rod (impactor) is described by the following set of equations:

$$\frac{\partial \sigma}{\partial z} = \rho_{\text{im}} \dot{v}, \tag{34}$$

$$\dot{\sigma} = E_{\text{im}} \frac{\partial v}{\partial z}, \tag{35}$$

where σ is the stress, v is the velocity, ρ_{im} and E_{im} are the density and Young's modulus of impactor's material, respectively.

The impact occurs at $t = 0$ at the point with the coordinate $z = 0$. The equation of motion of the contact zone, which is considered to be rigid and which is enclosed between the planes $z = \pm\tau_{\text{im}}$ (Fig. 1) is written as

$$2\tau_{\text{im}} F\rho \ddot{u}_y = 2N \frac{\partial u_y}{\partial z}\bigg|_{z=\tau_{\text{im}}} + 2Q|_{z=\tau_{\text{im}}} + F_{\text{cont}}, \tag{36}$$

where the value $(2N\partial u_y/\partial z)|_{z=\tau_{\text{im}}}$ takes the extension of beam's median surface into account.

The contact stress within the contact domain could be determined as

$$\sigma_{\text{cont}} = \frac{F_{\text{cont}}}{2\tau_{\text{im}} a} = E_\infty\big[1 - \nu_\varepsilon \ni^*_\gamma \big(\tau^\gamma_\varepsilon\big)\big] \frac{(\alpha - u_y)}{h}, \tag{37}$$

wherein α and u_y are, respectively, the displacements of the upper and lower layers of the beam of the hight h, which models the process of the interaction of the contact domain with the impactor, resulting in the generation of transverse deformation of the contact zone equal to $(\alpha - u_y)h^{-1}$, γ $(0 < \gamma < 1)$ is the fractional parameter,

$$\left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^\gamma = \frac{E_0}{E_\infty}, \tag{38}$$

$$\ni^*_\gamma \big(\tau^\gamma_i\big) Z(t) = \int_0^t \ni_\gamma \left(\frac{t - t'}{\tau_i}\right) Z\big(t'\big)\, \mathrm{d}t' \quad (i = \varepsilon, \sigma), \tag{39}$$

$$\ni_\gamma \left(\frac{t}{\tau_i}\right) = \frac{t^{\gamma-1}}{\tau_i^\gamma} \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau_i)^{\gamma n}}{\Gamma[\gamma(n+1)]}, \tag{40}$$

$\Gamma[\gamma(n+1)]$ is the Gamma-function, $\ni_\gamma\ (t/\tau_i)$ is the Rabotnov fractional exponential function, which at $\gamma = 1$ becomes the ordinary exponent, while the operator $\ni_\gamma\ (\tau_i)$ becomes the operator $\ni_1\ (\tau_i)$. When $\gamma \to 0$, function $\ni_\gamma\ (t/\tau_i) \to \delta(t)$, and σ_{cont} turns out to be elastic and equal to

$$\sigma_{\text{cont}} = E_0(\alpha - u_y)/h. \tag{41}$$

It is seen that the fractional parameter γ is the structural parameter allowing one to control the viscosity of the target material from the pure elastic case at $\gamma = 0$ to conventional viscous state at $\gamma = 1$.

From (37) it follows that the contact force can be calculated as

$$F_{\text{cont}} = \hat{E}_\infty\big[1 - \nu_\varepsilon \ni^*_\gamma \big(\tau^\gamma_\varepsilon\big)\big](\alpha - u_y), \tag{42}$$

where $\hat{E}_\infty = E_\infty \frac{2\tau_{\text{im}} a}{h}$ is the non-relaxed elastic modulus of the fractional derivative standard linear solid model valid within the contact domain, while the relative displacement within the contact zone is defined as

$$\alpha - u_y = \hat{E}_\infty^{-1}\left[1 + \nu_\sigma \ni_\gamma^* \left(\tau_\sigma^\gamma\right)\right] F_{\text{cont}}. \tag{43}$$

Equation (36) is subjected to the initial conditions

$$\alpha|_{t=0} = u_y|_{t=0} = \dot{u}_y|_{t=0} = 0, \quad \dot{\alpha}|_{t=0} = V_0. \tag{44}$$

If we substitute formula (40) into (39) and consider relationship for the fractional integral

$$I^\gamma Z(t) = \int_0^t \frac{(t-t')^{\gamma-1}}{\Gamma(\gamma)} Z\left(t'\right) \mathrm{d}t' \quad (0 < \gamma \le 1), \tag{45}$$

then as a result we obtain

$$\ni_\gamma^* \left(\tau_i^\gamma\right) Z(t) = \sum_{n=0}^{\infty} (-1)^n \tau_i^{-\gamma(n+1)} I^{\gamma(n+1)} Z(t). \tag{46}$$

If the series on the right-hand side of (46) is interpreted as an infinite decreasing geometric progression with the denominator $q = -I^\gamma \tau_i^{-\gamma}$ and the first term $a_1 = I^\gamma \tau_i^{-\gamma}$, then the sum of this series could be written in the form

$$\ni_\gamma^* \left(\tau_i^\gamma\right) Z(t) = \frac{I^\gamma \tau_i^{-\gamma}}{1 - (-I^\gamma \tau_i^{-\gamma})} Z(t). \tag{47}$$

Introducing into consideration the fractional derivative

$$D^\gamma Z(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{(t-t')^{-\gamma}}{\Gamma(1-\gamma)} Z\left(t'\right) \mathrm{d}t' \quad (0 < \gamma \le 1), \tag{48}$$

multiplying the numerator and denominator of the fraction on the right-hand side of formula (47) by $D^\gamma \tau_i^\gamma$, and accounting for $D^\gamma I^\gamma = I^\gamma D^\gamma = 1$, we have

$$\ni_\gamma^* \left(\tau_i^\gamma\right) Z(t) = \frac{1}{1 + \tau_i^\gamma D^\gamma} Z(t), \tag{49}$$

i.e.

$$\ni_\gamma^* \left(\tau_i^\gamma\right) = \frac{1}{1 + \tau_i^\gamma D^\gamma} \tag{50}$$

is the dimensionless Rabotnov operator (Rossikhin and Shitikova 2014).

Following (50), we write

$$\ni_\gamma^* \left(\tau_\varepsilon^\gamma\right) \ni_\gamma^* \left(\tau_\sigma^\gamma\right) = \frac{1}{(1 + \tau_\varepsilon^\gamma D^\gamma)(1 + \tau_\sigma^\gamma D^\gamma)} = \frac{A_2}{1 + \tau_\varepsilon^\gamma D^\gamma} + \frac{B_2}{1 + \tau_\sigma^\gamma D^\gamma},$$

where

$$A_2 = \frac{\tau_\varepsilon^\gamma}{\tau_\varepsilon^\gamma - \tau_\sigma^\gamma}, \qquad B_2 = -\frac{\tau_\sigma^\gamma}{\tau_\varepsilon^\gamma - \tau_\sigma^\gamma},$$

whence it follows that

$$\ni_\gamma^*\left(\tau_\varepsilon^\gamma\right)\ni_\gamma^*\left(\tau_\sigma^\gamma\right)=\frac{\tau_\varepsilon^\gamma\ni_\gamma^*\left(\tau_\varepsilon^\gamma\right)-\tau_\sigma^\gamma\ni_\gamma^*\left(\tau_\sigma^\gamma\right)}{\tau_\varepsilon^\gamma-\tau_\sigma^\gamma}. \tag{51}$$

Comparing formulas (17) and (18) with (50) and (51), we see that the first formulas are obtained from the second at $\gamma = 1$.

The Rabotnov fractional exponential function (Rabotnov 1948) is the second function after the exponential function, the resolvent kernels of which coincide (see formulas (42) and (43)). In order to prove the validity of formulas (42) and (43), it is sufficient to eliminate the value $\alpha - u_y$ from them, to utilise the theorem of multiplication (51), as well as relationship (38), and then as a result we obtain the identity.

Comparing operators in (42) and (43) with the corresponding operators in (7) and (19), we see their differences, although the material of the beam within and out of the contact zone is the same. Such a distinction is connected with the fact that during the process of impact decrosslinking of long molecules of the viscoelastic material occurs within the domain of the contact of the beam with the rod, resulting in more free displacements of molecules with respect to each other, and finally in the decrease of the beam material viscosity in the contact zone. Since the fractional parameter γ controls the magnitude of viscosity from its initial value at $\gamma = 1$ to the vanishing value at $\gamma = 0$, that is why such a substitution is quite justified.

3 Method of solution

To find the solution of the stated problem, two methods are used, namely, the ray method and Laplace transform technique. The ray method is applied for constructing an approximate solution within the elastic part of the beam from the surface of strong discontinuity up to the boundary of the contact region, as well as for finding the exact solution within the disturbed domain of the elastic rod. Within the contact domain, the Laplace transformation method is utilised to determine the contact force.

3.1 A ray method for a viscoelastic Timoshenko beam

To solve Eq. (36), it is necessary to find the values N and Q.

Under proposed assumptions concerning the contact domain, plane transient longitudinal and transverse shear waves (surfaces of strong discontinuity) propagate from the boundary of the contact zone during the process of impact. A certain function to be found could be represented in the form of a ray series

$$Z(z,t)=\sum_{\alpha=1}^{2}\sum_{k=0}^{\infty}\frac{1}{k!}[Z_{,(k)}]|_{t=z/G^{(\alpha)}}\left(t-\frac{z}{G^{(\alpha)}}\right)^k H\left(t-\frac{z}{G^{(\alpha)}}\right), \tag{52}$$

where $[Z_{,(k)}]=Z_{,(k)}^{+}-Z_{,(k)}^{-}=[\partial^k Z/\partial t^k]$ are the discontinuities in the kth order derivatives with respect to time t of the desired function $Z(z,t)$ on the wave surface, the superscript signs + and − denote that the given value is calculated immediately ahead of and behind the wave front, respectively, the index α labels the ordinal number of the wave, namely, $\alpha = 1$ for the longitudinal wave, and $\alpha = 2$ for the transverse wave, $H(t)$ is the Heaviside function, and $G^{(\alpha)}$ is the normal velocity of the surface of discontinuity.

To determine the coefficients of the ray series (52), it is necessary to differentiate the governing equations (1)–(6) k times with respect to time, take their difference on the different sides of the wave surface, and apply the condition of compatibility for the $(k+1)$-order discontinuities of the function $Z(x,t)$, which has the following form (Rossikhin and Shitikova 1995):

$$G\left[\frac{\partial Z_{,(k)}}{\partial z}\right] = -[Z_{,(k+1)}] + \frac{d[Z_{,(k)}]}{dt}, \tag{53}$$

where $\mathrm{d}/\mathrm{d}t$ is the complete time-derivative of the function $Z_{,(k)}(z,t)$ on the moving surface of discontinuity.

Since the process of impact is a transient process, first, it is possible to limit ourselves to the zeroth order terms of the ray series (52), and second, to neglect the waves reflected from the end face of the beam considering that they reach the contact zone after impactor's rebound from the beam.

Further we shall interpret a shock wave in the beam (surface of strong discontinuity) as a layer of small thickness δ, the head front of which arrives at a certain point M with the coordinate z at the moment of time t, while the back front of the shock layer reaches this point at the moment $t+\Delta t$. The desired values $Z(z,t)$ at the point M, such as velocity, generalized forces and deformations, during the time increment Δt change monotonically and uninterruptedly from the magnitude Z^- to the magnitude Z^+, in so doing within the layer, according to the condition of compatibility (53), the following relationships are fulfilled for the displacements and the angle of rotation:

$$\frac{\partial u_z}{\partial z} = -G^{-1} v_z, \tag{54}$$

$$\frac{\partial u_y}{\partial z} = -G^{-1} v_y, \tag{55}$$

$$\frac{\partial \psi}{\partial z} = -G^{-1} \Psi, \tag{56}$$

since $[u_z] = [u_y] = [\psi] = 0$, while for the values N, Q, and M the following condition of compatibility holds:

$$\frac{\partial Z}{\partial z} = -G^{-1}\frac{\partial Z}{\partial t} + G^{-1}\frac{dZ}{dt}. \tag{57}$$

Changing in (1)–(3), according to formula (44), the derivatives $\partial N/\partial z$, $\partial Q/\partial z$, and $\partial M/\partial z$ with the relationships $-G^{-1}(\partial N/\partial z - dN/dz)$, $-G^{-1}(\partial Q/\partial z - dQ/dz)$, and $-G^{-1}(\partial M/\partial z - dM/dz)$, respectively, integrating the resulting equations with respect to t from t to $t+\Delta t$, letting $\Delta t \to 0$, and considering that

$$\lim_{\Delta t\to 0}\int_t^{t+\Delta t} \frac{\mathrm{d}Z(z,t)}{\mathrm{d}t}\,\mathrm{d}t = 0,$$

we find

$$[N] = -\rho F G[v_z], \tag{58}$$

$$[Q] = -\rho F G[v_y], \tag{59}$$

$$[M] = \rho I G[\Psi]. \tag{60}$$

Substituting in (4)–(6) the derivatives $\partial u_z/\partial z$, $\partial u_y/\partial z$, and $\partial \psi/\partial z$ by the expressions $-G^{-1}v_z$, $-G^{-1}v_y$, and $-G^{-1}\Psi$ according to formulae (54)–(56), respectively, and writing them at the moments t and $t+\Delta t$, we obtain

$$N^- = -FE_\infty\left[G^{-1}v_z^- - \nu_\varepsilon \frac{1}{\tau_\varepsilon}\int_0^t e^{-\frac{t-t'}{\tau_\varepsilon}} G^{-1}v_z(t')\,\mathrm{d}t'\right], \tag{61}$$

$$N^+ = -FE_\infty\left[G^{-1}v_z^+ - \nu_\varepsilon \frac{1}{\tau_\varepsilon}\int_0^{t+\Delta t} e^{-\frac{t+\Delta t-t'}{\tau_\varepsilon}} G^{-1}v_z(t')\,\mathrm{d}t'\right], \tag{62}$$

$$Q^- = -KF\mu_\infty\left[G^{-1}v_y^- + \psi^- - n\frac{1}{t_\varepsilon}\int_0^t e^{-\frac{t-t'}{t_\varepsilon}}\left(G^{-1}v_y(t') + \psi(t')\right)\mathrm{d}t'\right], \tag{63}$$

$$Q^+ = -KF\mu_\infty\left[G^{-1}v_y^+ + \psi^+ - n\frac{1}{t_\varepsilon}\int_0^{t+\Delta t} e^{-\frac{t+\Delta t-t'}{t_\varepsilon}}\left(G^{-1}v_y(t') + \psi(t')\right)\mathrm{d}t'\right], \tag{64}$$

$$M^- = IE_\infty\left[G^{-1}\Psi^- - \nu_\varepsilon \frac{1}{\tau_\varepsilon}\int_0^t e^{-\frac{t-t'}{\tau_\varepsilon}} G^{-1}\Psi(t')\,\mathrm{d}t'\right], \tag{65}$$

$$M^+ = IE_\infty\left[G^{-1}\Psi^+ - \nu_\varepsilon \frac{1}{\tau_\varepsilon}\int_0^{t+\Delta t} e^{-\frac{t+\Delta t-t'}{\tau_\varepsilon}} G^{-1}\Psi(t')\,\mathrm{d}t'\right]. \tag{66}$$

Expanding the integrals in (62), (64), and (66) into the Taylor series with respect to the small parameter Δt and limiting ourselves to the zeroth and first order approximations, we have

$$\begin{aligned}&\int_0^{t+\Delta t} e^{-\frac{t+\Delta t-t'}{\tau_\varepsilon}} v_z(t')\,\mathrm{d}t'\\ &\quad= \int_0^t e^{-\frac{t-t'}{\tau_\varepsilon}} v_z(t')\,\mathrm{d}t' + v_z(t)\Delta t - \Delta t\frac{1}{\tau_\varepsilon}\int_0^t e^{-\frac{t-t'}{\tau_\varepsilon}} v_z(t')\,\mathrm{d}t',\end{aligned} \tag{67}$$

$$\begin{aligned}&\int_0^{t+\Delta t} e^{-\frac{t+\Delta t-t'}{t_\varepsilon}}\left[G^{-1}v_y(t') + \psi(t')\right]\mathrm{d}t'\\ &\quad= \int_0^t e^{-\frac{t-t'}{t_\varepsilon}}\left[G^{-1}v_y(t') + \psi(t')\right]\mathrm{d}t' + \left[G^{-1}v_y(t') + \psi(t')\right]\Delta t\\ &\qquad - \Delta t\frac{1}{t_\varepsilon}\int_0^t e^{-\frac{t-t'}{t_\varepsilon}}\left[G^{-1}v_y(t') + \psi(t')\right]\mathrm{d}t',\end{aligned} \tag{68}$$

$$\begin{aligned}&\int_0^{t+\Delta t} e^{-\frac{t+\Delta t-t'}{\tau_\varepsilon}} \psi(t')\,\mathrm{d}t'\\ &\quad= \int_0^t e^{-\frac{t-t'}{\tau_\varepsilon}} \psi(t')\,\mathrm{d}t' + \psi(t)\Delta t - \Delta t\frac{1}{\tau_\varepsilon}\int_0^t e^{-\frac{t-t'}{\tau_\varepsilon}} \psi(t')\,\mathrm{d}t'.\end{aligned} \tag{69}$$

Subtracting (61), (63), and (65), respectively, from (62), (64), and (66), accounting for (67)–(69), and letting Δt to zero, we find

$$[N] = -FE_\infty G^{-1}[v_z], \tag{70}$$

$$[Q] = -KF\mu_\infty G^{-1}[v_y], \tag{71}$$

$$[M] = IE_\infty G^{-1}[\Psi]. \tag{72}$$

From relationships (58)–(60) and (70)–(72) it is possible to find the velocities of two types of transient waves:

– longitudinal–flexural wave

$$G_\infty^{(1)} = \left(\frac{E_\infty}{\rho}\right)^{1/2}, \tag{73}$$

– and shear wave

$$G_\infty^{(2)} = \left(\frac{K\mu_\infty}{\rho}\right)^{1/2}. \tag{74}$$

Substituting the found velocities (73) and (74) in formulae (58)–(60) and considering, as it has been already mentioned, the zeroth order terms of the ray series, we have

$$N = -\rho F G_\infty^{(1)} v_z, \tag{75}$$

$$Q = -\rho F G_\infty^{(2)} v_y, \tag{76}$$

$$M = \rho I G_\infty^{(1)} \Psi. \tag{77}$$

Note that relationships (75)–(77) do not differ from those for an elastic beam, since at the moment of impact a viscoelastic medium behaves as an elastic medium with the unrelaxed elastic modulus.

3.2 Ray method for the three-dimensional approach for a viscoelastic beam

The Timoshenko equations (1)–(6) possess one essential drawback, namely, they involve the shear coefficient K which is not determined experimentally and depends on geometric parameters of the cross-section of a beam.

In a series of papers (Rossikhin and Shitikova 2007a, 2008, 2012) and in a monograph (Rossikhin and Shitikova 2011), the authors of this paper have developed a brand new approach for obtaining hyperbolic sets of equations describing the dynamic behaviour of elastic, viscoelastic and thermoelastic bodies subjected to transient excitations. This approach is based on three-dimensional dynamic equations of the medium, which a thin body under consideration is made of, namely, a plate, a shell, a rod of solid cross-section, a thin-walled beam of open or closed profile, and so on. As this takes place, the theory of discontinuities is utilised, making it possible to obtain recurrent equations of the ray method and allowing one to determine the discontinuities in the desired values and their time-derivatives of arbitrary order, and then to construct the ray series for the desired values, which in its turn, enables one to solve boundary-value dynamic problems dealing with transient excitations upon thin bodies. The developed approach eliminates any possibility of appearance of any additional coefficients like the shear coefficients, which are presented in all Timoshenko-type theories, and it operates only with constants of the material which the thin body is made of. Dynamic equations obtained within this approach take rotary inertia and transverse shear deformations, as well as transverse compression effect into account.

Here we will utilise this approach for determining the zeroth order coefficients of the ray series for the desired functions and the correct velocities of transient waves of shear for a viscoelastic beam of solid cross-section.

For this purpose, we write the stress tensor for a viscoelastic medium, considering formulae (32) and (33), as

$$\sigma_{ij} = \lambda_\infty\left[1 + n_1 \ni_1^*(t_\sigma)\right]u_{l,l}\delta_{ij} + \mu_\infty\left[1 - n \ni_1^*(t_\sigma)\right](u_{i,j} + u_{j,i}), \tag{78}$$

where summation is carried out over two repeated indices, an index after a comma labels the derivative with respect to the corresponding coordinate, σ_{ij} and u_i are the stress tensor and displacement vector components, respectively, $x = x_1$, $y = x_2$, $z = x_3$, and δ_{ij} is the Kronecker symbol ($i, j = 1, 2, 3$).

Using the procedure applied above to deduce formulae (70)–(72), from relationship (78) we obtain

$$[\sigma_{ij}] = \lambda_\infty [u_{l,l}]\delta_{ij} + \mu_\infty\big([u_{i,j}] + [u_{j,i}]\big). \tag{79}$$

Considering the generalized conditions of compatibility (Rossikhin and Shitikova 2007a),

$$[u_{l,l}] = -G^{-1}[v_l]\nu_l + [u_{x,x}] + [u_{y,y}], \tag{80}$$

$$[u_{i,j}] = -G^{-1}[v_i]\nu_j + [u_{i,x}]k_j + [u_{i,y}]s_j, \tag{81}$$

distinct from the conditions of compatibility (49)–(51) since they take the transverse deformations into account, which is characteristic for beams, we rewrite relationship (79) in the form

$$\begin{aligned}[\sigma_{ij}] = {} & -\lambda_\infty G^{-1}[v_z]\delta_{ij} - \mu_\infty G^{-1}\big([v_i]\nu_j + [v_j]\nu_i\big) \\ & + \mu_\infty\big([u_{i,x}]k_j + [u_{j,x}]k_i + [u_{i,y}]s_j + [u_{j,y}]s_i\big) \\ & + \lambda_\infty\big([\epsilon_x] + [\epsilon_y]\big)\delta_{ij},\end{aligned} \tag{82}$$

where

$$[v_z] = [v_i]\nu_i, \qquad [\epsilon_x] = [u_{i,x}]k_i = [u_{x,x}], \qquad [\epsilon_y] = [u_{i,y}]s_i = [u_{y,y}].$$

The dynamic condition of compatibility

$$[\sigma_{ij}]\nu_j = -\rho G[v_i], \tag{83}$$

which is obtained from the equation of motion

$$[\sigma_{ij,j}] = \rho v_i, \tag{84}$$

should be added to (80) and (81).

Multiplying (81) sequentially by $k_i k_j$, $s_i s_j$, and $s_i k_j$ and neglecting the press of layers within the front of the surface of strong discontinuity in the direction of the vectors **k** and **s**, i.e. considering that

$$[\sigma_{ij}]k_i k_j = [\sigma_{ij}]s_i s_j = [\sigma_{ij}]s_i k_j = 0, \tag{85}$$

we have

$$[u_{y,x}] = [u_{x,y}] = 0, \tag{86}$$

$$(\lambda_\infty + 2\mu_\infty)[\epsilon_x] + \lambda_\infty[\epsilon_y] = \lambda_\infty G^{-1}[v_z], \tag{87}$$

$$\lambda_\infty[\epsilon_x] + (\lambda_\infty + 2\mu_\infty)[\epsilon_y] = \lambda_\infty G^{-1}[v_z]. \tag{88}$$

From (81) and (82) we find

$$[\epsilon_x] = [\epsilon_y] = G^{-1}\frac{\lambda_\infty}{2(\lambda_\infty + \mu_\infty)}[v_z], \tag{89}$$

and accounting for

$$\nu_\infty = \frac{\lambda_\infty}{2(\lambda_\infty + \mu_\infty)},$$

we have

$$[\epsilon_x] = [\epsilon_y] = \nu_\infty G^{-1}[v_z]. \tag{90}$$

Substituting (89) into (82) and multiplying the resulting equation by $\nu_i \nu_j$, we find

$$[\sigma_{ij}]\nu_i \nu_j = -\frac{\mu_\infty(3\lambda_\infty + 2\mu_\infty)}{\lambda_\infty + \mu_\infty} G^{-1}[v_z], \tag{91}$$

and considering that

$$E_\infty = \frac{\mu_\infty(3\lambda_\infty + 2\mu_\infty)}{\lambda_\infty + \mu_\infty},$$

we obtain

$$[\sigma_{ij}]\nu_i \nu_j = -E_\infty G^{-1}[v_z]. \tag{92}$$

Let us multiply (83) by ν_i, giving

$$[\sigma_{ij}]\nu_i \nu_j = -\rho G[v_z], \tag{93}$$

and eliminate the value $[\sigma_{ij}]\nu_i \nu_j$ from (92) and (93). As a result we obtain the velocity of the transient longitudinal–flexural wave (73).

Multiplying (82) and (83) by $\nu_j k_i$ and k_i, and then by $\nu_j s_i$ and s_i, respectively, we have

$$[\sigma_{ij}]\nu_j k_i = -\mu_\infty G^{-1}[v_x], \tag{94}$$

$$[\sigma_{ij}]\nu_j k_i = -\rho G[v_x], \tag{95}$$

and

$$[\sigma_{ij}]\nu_j s_i = -\mu_\infty G^{-1}[v_y], \tag{96}$$

$$[\sigma_{ij}]\nu_j k_i = -\rho G[v_y]. \tag{97}$$

Eliminating the value $[\sigma_{ij}]\nu_j k_i$ from (94) and (95) and $[\sigma_{ij}]\nu_j s_i$ from (96) and (97), we obtain the velocity of the transient shear wave

$$G_\infty^{(2)} = \left(\frac{\mu_\infty}{\rho}\right)^{1/2}, \tag{98}$$

on which the shear takes place both along the x and y axes.

It is seen that, despite (74), formula (98) is free from the coefficient K.

Now we will estimate the value $(2N\partial u_y/\partial z)|_{z=\tau_{\rm im}}$ in (36), which is responsible for stretching of the median plane of the beam.

Since in further treatment we will utilise one-term ray expansions, relationships (90), accounting for (54), take the form

$$u_{y,y} = u_{x,x} = \nu_\infty G_\infty^{(1)^{-1}} v_z = -\nu_\infty u_{z,z}, \tag{99}$$

or

$$v_z = \nu_\infty^{-1} G_\infty^{(1)} u_{y,y}. \tag{100}$$

Here velocity G has been substituted by velocity $G_\infty^{(1)}$, since formulae (99) and (100) are valid only on the longitudinal–flexural waves.

Considering that $u_{y,y}$ at $z = \tau_{\text{im}}$ is the deformation of the buffer of the hight h which models the transverse deformation of the contact domain during impact, formula (100) can be rewritten in the form

$$v_z = \nu_\infty^{-1} G_\infty^{(1)} (\alpha - u_y)/h. \tag{101}$$

Then the value $2N\partial u_y/\partial z$ at $z = \tau_{\text{im}}$, taking into account (101), (55), and (75), takes the form

$$2N \frac{\partial u_y}{\partial z}\bigg|_{z=\tau_{\text{im}}} = e(\alpha - u_y) v_y, \tag{102}$$

where $e = 2\rho F {G_\infty^{(1)}}^2 (G_\infty^{(2)} \nu_\infty h)^{-1}$.

From the initial conditions (44) it is evident that the value defined by formula (102) has the second order of smallness, it could be neglected with respect to other values in (36).

As for the value Q, which also enters in (36), for its calculation it is sufficient to integrate (97) from the right and from the left over the cross-sectional area of the beam and to drop the brackets. As a result we obtain (76) wherein $G_\infty^{(2)}$ is defined by (98).

If we neglect the first term on the right-hand side of (36), responsible for the stretching of the middle plane of the beam, and substitute the value Q by expression (76), then as a result we obtain the final form of the equation of motion of the contact domain:

$$m\ddot{u}_y + mB\dot{u}_y = F_{\text{cont}}, \tag{103}$$

where $B = \tau_{\text{im}}^{-1} G_\infty^{(2)}$ and $m = 2\tau_{\text{im}}\rho F$ is the mass of the contact zone.

3.3 Ray method for the elastic rod

At the moment of impact of a projectile (rod) against a target (beam), shock waves are generated not only in the beam but in the rod as well: a longitudinal shock wave propagates along the rod with velocity G_{im}.

Using the same reasoning for determining the dynamic conditions of compatibility as above for the viscoelastic beam, we find

$$[\sigma] = -\rho_{\text{im}} G_{\text{im}} [v], \qquad -G_{\text{im}}[\sigma] = E_{\text{im}}[v], \tag{104}$$

whence it follows that

$$G_{\text{im}} = \sqrt{\frac{E_{\text{im}}}{\rho_{\text{im}}}}. \tag{105}$$

Behind the front of this wave (a surface of the strong discontinuity), the relationships for the stress σ^- and velocity v^- could be obtained using the ray series (Rossikhin and Shitikova 2007b) as follows:

$$\sigma^- = -\sum_{k=0}^{\infty} \frac{1}{k!} [\sigma_{,(k)}] \left(t - \frac{z}{G_{\text{im}}} \right)^k, \tag{106}$$

$$v^- = V_0 - \sum_{k=0}^{\infty} \frac{1}{k!}[v_{,(k)}]\left(t - \frac{z}{G_{\mathrm{im}}}\right)^k, \tag{107}$$

where $[\sigma_{,(k)}] = [\partial^k \sigma / \partial t^k]$ and $[v_{,(k)}] = [\partial^k v / \partial t^k]$.

Considering that the discontinuities in the elastic rod remain constant during the process of the wave propagation, and using the condition of compatibility

$$G_{\mathrm{im}}\left[\frac{\partial Z_{,(k-1)}}{\partial z}\right] = -[Z_{,(k)}],$$

which is obtained from Eq. (57) by substitution of the function Z with $Z_{,(k)} = \partial^k Z / \partial t^k$, we have

$$\left[\frac{\partial \sigma_{,(k-1)}}{\partial z}\right] = -G_{\mathrm{im}}^{-1}[\sigma_{,(k)}]. \tag{108}$$

Accounting for (108), the equation of motion on the wave surface is written in the form

$$[\sigma_{,(k)}] = -\rho_{\mathrm{im}} G_{\mathrm{im}}[v_{,(k)}]. \tag{109}$$

Substituting (109) into (106) yields

$$\sigma^- = \rho_{\mathrm{im}} G_{\mathrm{im}} \sum_{k=0}^{\infty} \frac{1}{k!}[v_{,(k)}]\left(t - \frac{z}{G_{\mathrm{im}}}\right)^k. \tag{110}$$

Comparing relationships (110) and (107), we obtain

$$\sigma^- = \rho_{\mathrm{im}} G_{\mathrm{im}}\left(V_0 - v^-\right). \tag{111}$$

At $y = 0$, expression (111) takes the form

$$\sigma_{\mathrm{cont}} = \rho_{\mathrm{im}} G_{\mathrm{im}}(V_0 - v_y - \dot{\alpha}), \tag{112}$$

where $\sigma_{\mathrm{cont}} = \sigma^-|_{y=0}$ is the contact stress.

Using formula (111), it is possible to find the contact force

$$F_{\mathrm{cont}} = b(V_0 - \dot{u}_y - \dot{\alpha}), \tag{113}$$

where $b = 2\tau_{\mathrm{im}} a \rho_{\mathrm{im}} G_{\mathrm{im}}$.

The ray series (106) and (107) are the Taylor series in the vicinity of the wave front $z = G_{\mathrm{im}} t$, and that is why they converge at least for the case $G_{\mathrm{im}} t - z < 1$, which could be realised in the case of short-time impact process, since in this case a transient wave cannot propagate far from the contact zone.

3.4 Solution of Eq. (103) by the Laplace transform technique

Let us write Eq. (103) as well as relationships (42) and (113) in the Laplace domain as

$$m p u_y (p + B) = \bar{F}_{\mathrm{cont}}, \tag{114}$$

$$\bar{F}_{\mathrm{cont}} = \hat{E}_{\infty}\left[1 - \nu_{\varepsilon} \overline{\ni_{\gamma}^{*}\left(\tau_{\varepsilon}^{\gamma}\right)}\right](\bar{\alpha} - \bar{u}_y), \tag{115}$$

$$\bar{F}_{\text{cont}} = b\left(\frac{V_0}{p} - p\bar{\alpha} - p\bar{u}_y\right), \tag{116}$$

where a bar over a function denotes the Laplace transform of the corresponding function, and p is a Laplace variable.

In order to find the Laplace transform of the Rabotnov fractional operator $\overline{\ni_\gamma^* (\tau_\varepsilon^\gamma) Z(t)}$, we utilise formulae (39) and (40). As a result we obtain

$$\overline{\ni_\gamma^* \left(\tau_\varepsilon^\gamma\right) Z(t)} = \overline{\ni_\gamma^* \left(\tau_\varepsilon^\gamma\right)}\,\overline{Z(t)} = \sum_{n=0}^{\infty} (-1)^n \tau_\varepsilon^{-\gamma(n+1)} p^{-\gamma(n+1)} \bar{Z}(p). \tag{117}$$

If we interpret the series in (117) as an infinitely decreasing geometric progression with the denominator $q = -\tau_\varepsilon^{-\gamma} p^{-\gamma}$ and the first term $a_1 = \tau_\varepsilon^{-\gamma} p^{-\gamma}$, then the sum of this series could be written in the form

$$\sum_{n=0}^{\infty} (-1)^n \tau_\varepsilon^{-\gamma(n+1)} p^{-\gamma(n+1)} = \frac{\tau_\varepsilon^{-\gamma} p^{-\gamma}}{1 - (-\tau_\varepsilon^{-\gamma} p^{-\gamma})} = \frac{1}{1 + (p\tau_\varepsilon)^\gamma}. \tag{118}$$

Considering (118), we have

$$\overline{\ni_\gamma^* \left(\tau_\varepsilon^\gamma\right) Z(t)} = \frac{1}{1 + (p\tau_\varepsilon)^\gamma} \bar{Z}(p), \tag{119}$$

and hence

$$\bar{F}_{\text{cont}} = \hat{E}_\infty \left[1 - \frac{\nu_\varepsilon}{1 + (p\tau_\varepsilon)^\gamma}\right] (\bar{\alpha} - \bar{u}_y),$$

or taking into account (38) and the relationship $1 - \nu_\varepsilon = \hat{E}_0 \hat{E}_\infty^{-1}$, we find

$$\bar{F}_{\text{cont}} = \hat{E}_0 \frac{1 + (p\tau_\sigma)^\gamma}{1 + (p\tau_\varepsilon)^\gamma} (\bar{\alpha} - \bar{u}_y), \tag{120}$$

where $\hat{E}_0 = E_0 \frac{2\tau_{\text{im}} a}{h}$ is the relaxed elastic modulus of the fractional derivative standard linear solid model valid within the contact domain.

From relationships (114) and (120) we obtain

$$\bar{\alpha}(p) = \frac{V_0}{p^2} - \left[\frac{m}{b}(p + B) + 1\right] \bar{u}_y(p). \tag{121}$$

Now eliminating $\bar{F}_{\text{cont}}$ from (114) and (120) and considering (121), we obtain

$$\bar{w}(p) = \frac{V_0 \Omega_\infty^2 (\tau_\sigma^{-\gamma} + p^\gamma)}{p^2 f_\gamma(p)}, \tag{122}$$

where $\Omega_\infty^2 = \hat{E}_\infty m^{-1}$, and

$$f_\gamma(p) = p^{2+\gamma} + \tau_\varepsilon^{-\gamma} p^2 + \left(B + \hat{E}_\infty b^{-1}\right) p^{1+\gamma} + \left(B + \hat{E}_0 b^{-1}\right) \tau_\varepsilon^{-\gamma} p + \hat{E}_\infty \left(B b^{-1} + m^{-1}\right) p^\gamma + \hat{E}_0 \left(B b^{-1} + m^{-1}\right) \tau_\varepsilon^{-\gamma}. \tag{123}$$

Substituting formulas (121) and (122) into (116) yields

$$\bar{F}_{\text{cont}}(p) = \frac{V_0 \hat{E}_\infty (p+B)(\tau_\sigma^{-\gamma} + p^\gamma)}{p f_\gamma(p)}. \tag{124}$$

Besides, it is possible to find the value $\bar{\alpha}(p)$, if we exclude the value $\bar{w}(p)$ defined by (122) from Eq. (121). As a result we obtain

$$\bar{\alpha}(p) = \frac{V_0}{p^2}\left\{1 - \left[\frac{m}{b}(p+B) + 1\right]\frac{\Omega_\infty^2(\tau_\sigma^{-\gamma} + p^\gamma)}{f_\gamma(p)}\right\}. \tag{125}$$

Now we will carry out the inverse transformation of formula (124). For this purpose, first we will investigate the roots of the characteristic equation

$$f_\gamma(p) = 0. \tag{126}$$

Let us multiply Eq. (126) by τ_ε^γ, represent p in the geometric form

$$p = re^{\mathrm{i}\psi}, \tag{127}$$

and introduce a new variable $x = (r\tau_\varepsilon)^\gamma$. As a result, Eq. (126) can be rewritten in the form

$$\begin{aligned} &r^2\left[xe^{i(2+\gamma)\psi} + e^{2i\psi}\right] + r\left[\left(B + \hat{E}_\infty b^{-1}\right)xe^{i(1+\gamma)\psi} + \left(B + \hat{E}_0 b^{-1}\right)e^{i\psi}\right] \\ &\quad + \left(Bb^{-1} + 2m^{-1}\right)\left(\hat{E}_\infty x e^{i\gamma\psi} + \hat{E}_0\right) = 0. \end{aligned} \tag{128}$$

Separating the real and imaginary parts in (128), we have

$$r^2 a_1 + r a_2 + a_3 = 0, \tag{129}$$

$$r^2 b_1 + r b_2 + b_3 = 0, \tag{130}$$

where

$$\begin{aligned} a_1 &= \cos 2\psi + x\cos(2+\gamma)\psi, \\ b_1 &= \sin 2\psi + x\sin(2+\gamma)\psi, \\ a_2 &= \left(B + \hat{E}_0 b^{-1}\right)\left[\cos\psi + x\left(B + \hat{E}_\infty b^{-1}\right)\cos(1+\gamma)\psi\right], \\ b_2 &= \left(B + \hat{E}_0 b^{-1}\right)\left[\sin\psi + x\left(B + \hat{E}_\infty b^{-1}\right)\sin(1+\gamma)\psi\right], \\ a_3 &= \left(Bb^{-1} + 2m^{-1}\right)\left(\hat{E}_0 + x\hat{E}_\infty \cos\gamma\psi\right), \\ b_3 &= \left(Bb^{-1} + 2m^{-1}\right)x\hat{E}_\infty \sin\gamma\psi. \end{aligned}$$

First, we fix the angle $\frac{\pi}{2} \le \psi \le \pi$ in Eqs. (129) and (130), and then eliminate r^2. As a result we obtain

$$r = \frac{a_1 b_3 - a_3 b_1}{a_2 b_1 - a_1 b_2}. \tag{131}$$

Substituting (131) into (129) yields

$$(a_3 b_1 - a_1 b_3)^2 a_1 - (a_2 b_1 - a_1 b_2)(a_3 b_1 - a_1 b_3)a_2 + (a_2 b_1 - a_1 b_2)^2 a_3 = 0. \tag{132}$$

From Eq. (132) at each fixed angle ψ from the segment $\frac{\pi}{2} \le \psi \le \pi$, we find the values x_i $(i = 1, 2, \dots)$, and then we substitute the chosen ψ with the found magnitude of x_i in Eq. (131), which allows us to find the corresponding modulus r_i $(i = 1, 2, \dots)$. Knowing the values of x_i and r_i, it is possible to determine $(\tau_\varepsilon^\gamma)_i = x_i r_i^{-\gamma}$. The set of values involving the angle ψ, radii r_i, and parameters $(\tau_\varepsilon^\gamma)_i$ completely defines the roots of the characteristic equation (126).

In order to clarify the number of characteristic equation roots, we consider their asymptotic behaviour.

3.4.1 The case $\tau_\varepsilon^\gamma \to 0$

Suppose that $\tau_\varepsilon^\gamma \to 0$ $(\tau_\varepsilon^{-\gamma} \to \infty)$. In this case, the characteristic equation (126) takes the form

$$f_{\gamma 0}(p_0) = p_0^2 + \left(B + \hat{E}_0 b^{-1}\right) p_0 + \hat{E}_0\left(B b^{-1} + m^{-1}\right) = 0, \tag{133}$$

whence it follows that

$$p_{0i} = (p_0)_{1,2} = -\frac{1}{2}\left(B + \hat{E}_0 b^{-1}\right) \pm \frac{1}{2}\sqrt{\left(B - \hat{E}_0 b^{-1}\right)^2 - 8\hat{E}_0 m^{-1}}. \tag{134}$$

3.4.2 The case $\tau_\varepsilon^\gamma = \varepsilon$

Now we suppose that the relaxation time of the system is small, i.e. $\tau_\varepsilon^\gamma = \varepsilon$, where ε is a small value. We will seek the solution of the characteristic equation (126) in the form

$$p_i = p_{0i} + \varepsilon \chi_i, \tag{135}$$

where χ_i is an unknown function for now.

Substituting (135) into (126) and ignoring the values of the order higher than ε, we find

$$\chi_i = -\frac{f_{\gamma\infty}(p_{0i})}{f'_{\gamma 0}(p_{0i})}, \tag{136}$$

where $f'_\gamma(p)$ denotes the derivative of the function $f_\gamma(p)$ with respect to p,

$$f'_\gamma(p_{0i}) = 2p_{0i} + B + \hat{E}_0 b^{-1},$$

$$f_{\gamma\infty} = p_{0i}^{2+\gamma} + \left(B + \hat{E}_\infty b^{-1}\right) p_{0i}^{1+\gamma} + \hat{E}_\infty\left(B b^{-1} + 2m^{-1}\right) p_{0i}^{\gamma}.$$

3.4.3 The case $\tau_\varepsilon^\gamma \to \infty$

Suppose that $\tau_\varepsilon^\gamma \to \infty$ $(\tau_\varepsilon^{-\gamma} \to 0)$. In this case, the characteristic equation (126) takes the form

$$f_{\gamma\infty}(p_\infty) = p_\infty^{2+\gamma} + \left(B + \hat{E}_\infty b^{-1}\right) p_\infty^{1+\gamma} + \hat{E}_\infty\left(B b^{-1} + 2m^{-1}\right) p_\infty^{\gamma} = 0. \tag{137}$$

From Eq. (137) we find

$$p_{\infty i} = (p_\infty)_{1,2} = -\frac{1}{2}\left(B + \hat{E}_\infty b^{-1}\right) \pm \frac{1}{2}\sqrt{\left(B - \hat{E}_\infty b^{-1}\right)^2 - 8\hat{E}_\infty m^{-1}}. \tag{138}$$

3.4.4 The case $\tau_\varepsilon^{-\gamma} = \varepsilon$

Now we suppose that the relaxation time of the system is large, i.e. $\tau_\varepsilon^{-\gamma} = \varepsilon$, where ε is a small value. We will seek the solution of the characteristic equation (126) in the form

$$p_i = p_{\infty i} + \varepsilon \eta_i. \tag{139}$$

Substituting (139) into Eq. (126) and ignoring the values of the order higher than ε, we find

$$\eta_i = -\frac{f_{\gamma 0}(p_{\infty i})}{f'_{\gamma\infty}(p_{\infty i})}, \tag{140}$$

where

$$f_{\gamma 0}(p_{\infty i}) = p_{\infty i}^2 + \big(B + \hat{E}_0 b^{-1}\big) p_{\infty i} + \hat{E}_0\big(Bb^{-1} + 2m^{-1}\big),$$
$$f'_{\gamma\infty}(p_{\infty i}) = (2+\gamma) p_{\infty i}^{1+\gamma} + \big(B + \hat{E}_\infty b^{-1}\big)(1+\gamma) p_{\infty i}^{\gamma} + \big(Bb^{-1} + 2m^{-1}\big)\gamma p_{\infty i}^{\gamma - 1}.$$

On the ground of the above asymptotic formulas, it could be assumed that the characteristic equation (126) possesses two complex conjugate roots, which we will represent in the following form:

$$p_{1,2} = re^{\pm \mathrm{i}\psi} = -\kappa \pm \mathrm{i}\omega. \tag{141}$$

Further, it is convenient to rewrite $\bar{F}_{\mathrm{cont}}(p)$ defined by (124) in the form

$$\bar{F}_{\mathrm{cont}}(p) = \frac{1}{p} \bar{F}_0(p), \tag{142}$$

where

$$\begin{aligned} \bar{F}_0(p) &= V_0 \frac{g_\gamma(p)}{f_\gamma(p)}, \\ g_\gamma(p) &= \hat{E}_\infty p^{1+\gamma} + \hat{E}_0 \tau_\varepsilon^{-\gamma} p + \hat{E}_\infty B p^\gamma + \hat{E}_0 \tau_\varepsilon^{-\gamma} B. \end{aligned} \tag{143}$$

The function $F_0(t)$ in the time domain is governed by the Mellin–Fourier inversion formula

$$F_0(t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \bar{F}_0(p) e^{pt}\, \mathrm{d}p. \tag{144}$$

To calculate the integral (144), it is necessary to define all singular points of the complex function $\bar{F}_{\mathrm{cont}}(p)$. This multi-valued function possesses the branch points at $p = 0$ and $p = \infty$ and the simple poles at the same magnitudes of $p = p_k$ which make the denominator in Eq. (143) zero, i.e. they are the roots of the characteristic equation (126).

The inversion theorem is applicable to multi-valued functions possessing branch points only on the first sheet of the Riemann surface, i.e. when $0 < |\arg p| < \pi$. Thus a closed contour of integration should be chosen in the form presented in Fig. 2. Considering Jordan lemma and applying the main theorem of the theory of residues, we rewrite the integral (144) in the following form:

$$F_0(t) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \big[\bar{F}_0\big(se^{-\mathrm{i}\pi}\big) - \bar{F}_0\big(se^{\mathrm{i}\pi}\big)\big] e^{-st}\, \mathrm{d}s + \sum_k \mathrm{res}\big[\bar{F}_0(p_k) e^{p_k t}\big], \tag{145}$$

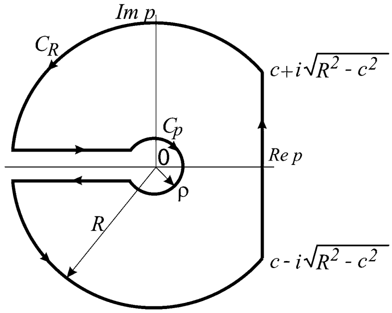

Fig. 2 Closed contour of integration

where the summation is carried out over all isolated singular points (poles).

Knowing the function $F_0(t)$, it is possible to determine the contact force $F_{\mathrm{cont}}(t)$ via the following formula:

$$F_{\mathrm{cont}}(t) = \int_0^t F_0(t')\,\mathrm{d}t'. \tag{146}$$

Since the roots of the characteristic equation (126) are complex conjugates and are defined by (141), Eq. (145) is reduced to

$$F_0(t) = A_0(t) + A\exp(-\kappa t)\cos(\omega t + \phi), \tag{147}$$

where

$$A_0(t) = \int_0^\infty B(s)e^{-st}\,\mathrm{d}s,$$

$$B(s) = \frac{(s-B)[Y\,\mathrm{Re}\,f_\gamma(se^{\mathrm{i}\pi}) - X\,\mathrm{Im}\,f_\gamma(se^{\mathrm{i}\pi})]V_0\pi^{-1}}{[\mathrm{Re}\,f_\gamma(se^{\mathrm{i}\pi})]^2 + [\mathrm{Im}\,f_\gamma(se^{\mathrm{i}\pi})]^2},$$

$$A_j = \frac{2V_0\sqrt{[N_1(p_j)]^2 + [N_2(p_j)]^2}}{[\mathrm{Re}\,f'_\gamma(p_j)]^2 + [\mathrm{Im}\,f'_\gamma(p_j)]^2}, \quad A_1 = A_2 = A,$$

$$\tan\phi_j = \frac{\mathrm{Re}\,f'_\gamma(p_j)\,\mathrm{Re}\,g_\gamma(p_j) + \mathrm{Im}\,f'_\gamma(p_j)\,\mathrm{Im}\,g_\gamma(p_j)}{\mathrm{Im}\,f'_\gamma(p_j)\,\mathrm{Re}\,g_\gamma(p_j) - \mathrm{Re}\,f'_\gamma(p_j)\,\mathrm{Im}\,g_\gamma(p_j)},$$

$$\tan\phi_1 = -\tan\phi_2 = \tan\phi,$$

$$\begin{aligned}
\mathrm{Re}\,f_\gamma\big(se^{\mathrm{i}\pi}\big) &= \tau_\varepsilon^{-\gamma}\big\{s^2\big[(s\tau_\varepsilon)^\gamma\cos(2+\gamma)\pi + 1\big] + s\big[(s\tau_\varepsilon)^\gamma\big(B + \hat{E}_\infty b^{-1}\big)\cos(1+\gamma)\pi \\
&\quad - \big(B + \hat{E}_0 b^{-1}\big)\big] + \big(Bb^{-1} + 2m^{-1}\big)\big[\hat{E}_\infty(s\tau_\varepsilon)^\gamma\cos\gamma\pi + \hat{E}_0\big]\big\} \\
&= \mathrm{Re}\,f_\gamma\big(se^{-\mathrm{i}\pi}\big),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Im}\,f_\gamma\big(se^{\mathrm{i}\pi}\big) &= \tau_\varepsilon^{-\gamma}\big[s^2(s\tau_\varepsilon)^\gamma\sin(2+\gamma)\pi + s(s\tau_\varepsilon)^\gamma\big(B + \hat{E}_\infty b^{-1}\big)\sin(1+\gamma)\pi \\
&\quad + \big(Bb^{-1} + 2m^{-1}\big)\hat{E}_\infty(s\tau_\varepsilon)^\gamma\sin\gamma\pi\big] = -\mathrm{Im}\,f_\gamma\big(se^{-\mathrm{i}\pi}\big),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}\,f'_\gamma(p_1) &= (2+\gamma)r^{1+\gamma}\cos(1+\gamma)\psi + 2r\tau_\varepsilon^{-\gamma}\cos\psi + (1+\gamma)\big(B + \hat{E}_\infty b^{-1}\big)r\cos\gamma\psi \\
&\quad + \gamma\hat{E}_\infty\big(Bb^{-1} + 2m^{-1}\big)r^{\gamma-1}\cos(\gamma-1)\psi + \big(B + \hat{E}_0 b^{-1}\big)\tau_\varepsilon^{-\gamma} = \mathrm{Re}\,f'_\gamma(p_2),
\end{aligned}$$

$$\mathrm{Im}\,f'_\gamma(p_1) = (2+\gamma)r^{1+\gamma}\sin(1+\gamma)\psi + 2r\tau_\varepsilon^{-\gamma}\sin\psi + (1+\gamma)\big(B + \hat{E}_\infty b^{-1}\big)r\sin\gamma\psi$$

$$+\gamma\hat{E}_\infty\big(Bb^{-1}+2m^{-1}\big)r^{\gamma-1}\sin(\gamma-1)\psi=-\operatorname{Im}f'_\gamma(p_2),$$
$$X=\hat{E}_0\tau_\varepsilon^{-\gamma}+\hat{E}_\infty s^\gamma\cos\pi\gamma,\qquad Y=\hat{E}_\infty s^\gamma\sin\pi\gamma,$$
$$N_1(p_j)=\operatorname{Re}f'_\gamma(p_j)\operatorname{Re}g_\gamma(p_j)+\operatorname{Im}f'_\gamma(p_j)\operatorname{Im}g_\gamma(p_j),\qquad N_1(p_1)=N_1(p_2),$$
$$N_2(p_j)=\operatorname{Im}f'_\gamma(p_j)\operatorname{Re}g_\gamma(p_j)-\operatorname{Re}f'_\gamma(p_j)\operatorname{Im}g_\gamma(p_j),\qquad N_2(p_1)=-N_2(p_2),$$
$$\operatorname{Re}g_\gamma(p_1)=\hat{E}_\infty r^{1+\gamma}\cos(1+\gamma)\psi+\hat{E}_0\tau_\varepsilon^{-\gamma}r\cos\psi+B\hat{E}_\infty r^\gamma\cos\gamma\psi$$
$$+B\hat{E}_0\tau_\varepsilon^{-\gamma}=\operatorname{Re}g_\gamma(p_2),$$
$$\operatorname{Im}g_\gamma(p_1)=\hat{E}_\infty r^{1+\gamma}\sin(1+\gamma)\psi+\hat{E}_0\tau_\varepsilon^{-\gamma}r\sin\psi+B\hat{E}_\infty r^\gamma\sin\gamma\psi=-\operatorname{Im}g_\gamma(p_2).$$

The first term in Eq. (147) defines the drift of the equilibrium position, while the second term governs damping vibrations around the drifting equilibrium position.

According to Eq. (146), to determine the function $F_{\text{cont}}(t)$, it is needed to integrate the function (147) with respect to t from 0 to t. As a result we obtain

$$F_{\text{cont}}(t)=\int_0^\infty B(s)\big(1-e^{-st}\big)\,\mathrm{d}s+\frac{A}{\kappa^2+\omega^2}\big\{\kappa\sin\phi+\omega\cos\phi$$
$$-e^{-\kappa t}\big[\kappa\sin(\omega t+\phi)+\omega\cos(\omega t+\phi)\big]\big\},\tag{148}$$

or

$$F_{\text{cont}}(t)=\int_0^\infty B(s)\big(1-e^{-st}\big)\,\mathrm{d}s$$
$$+\frac{A}{\sqrt{\kappa^2+\omega^2}}\big[\sin(\phi+\phi_0)-e^{-\kappa t}\sin(\omega t+\phi+\phi_0)\big],\tag{149}$$

where

$$\tan\phi_0=\frac{\omega}{\kappa}.$$

The dimensionless time $t^*=\omega t$ dependence of the dimensionless contact force $F^*_{\text{cont}}=F_{\text{cont}}\sqrt{\kappa^2+\omega^2}/A$ is presented in Fig. 3 for different magnitudes of the fractional parameter γ which are indicated by figures near the corresponding curves, wherein $\gamma=0$ and 1 are in compliance with a pure elastic case and conventional viscoelastic case, respectively. Figure 3 shows that a decrease in the fractional parameter results in the decrease of both the maximum of the contact force and the duration of contact.

4 Conclusion

The problem on low-velocity impact of a long thin elastic rod with a flat end upon an infinite viscoelastic beam has been considered. To construct the solution outside the contact zone, two approaches have been used. The first one is based on the hyperbolic set of Timoshenko-type equations describing the dynamic behaviour of a viscoelastic beam, the damping features of which are defined by a standard linear solid model. The second approach is based on three-dimensional dynamic equations describing the behaviour of a viscoelastic medium, the properties of which are governed by the standard linear solid model with ordinary derivatives.

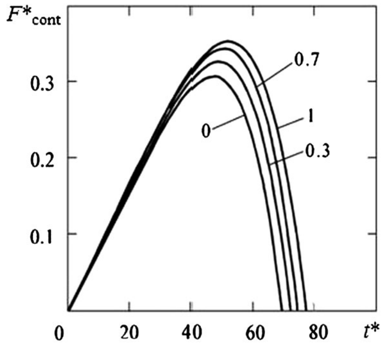

Fig. 3 Dimensionless time dependence of the dimensionless contact force

At the moment of impact, shock waves (surfaces of strong discontinuity) are generated both in the impactor and target, the influence of which on the contact domain has been analysed by the theory of discontinuities. It has been considered that the contact zone moves like a rigid whole under the action of the contact force and longitudinal and transverse forces applied to the boundary of the contact region, which were obtained on the basis of one-term ray expansions. During the impact process, decrosslinking occurs within the domain of the contact of the beam with the rod, resulting in more free displacements of molecules with respect to each other, and finally in the decrease of the beam material viscosity in the contact zone. This circumstance has allowed us to describe the behaviour of the beam material within the contact domain by the standard linear solid model involving fractional derivatives because the variation in the fractional parameter (the order of the fractional derivative) has enabled us to control the viscosity of the beam material.

Both approaches lead to the same results, the only difference being that within the first approach the velocity of the transient shear wave involves a certain shear coefficient which is not determined experimentally and depends on geometric form of beams' cross-section, while within the second approach this velocity is free from the shear coefficient and coincides with that for the shear wave in 3D medium. From the physical point of view, the second approach is preferable to the first one. Moreover, the second approach takes the rotary inertia, transverse shear deformations along both transverse axes of the beam, i.e. it allows one to consider the changes in the beam thickness.

Due to the short duration of contact interaction, the reflected waves were not taken into account, since it was assumed that the impactor bounced from the target before the reflected waves had time to reach the place of contact. Finally, the contact force has been determined analytically via the Laplace transform technique.

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Appendix

In the case when a mathematical pendulum is used for experimental measurement of internal friction in a beam, first of all operators $\widetilde{\mu}$ and $\widetilde{\mu}^{-1}$ can be determined, while the other operators are expressed in terms of them and the constant K_∞.

Thus, when operator $\widetilde{K} = K_\infty$ and operator $\tilde{\mu}$ is known and defined by

$$\tilde{\mu} = \mu_\infty\left[1 - \nu_\mu^\varepsilon \ni_1^* \left(\tau_\mu^\varepsilon\right)\right], \tag{150}$$

one needs to find the operator $\widetilde{E}$.

In this case, we will utilise the formula

$$\widetilde{E} = \frac{9K_\infty \tilde{\mu}}{3K_\infty + \tilde{\mu}}. \tag{151}$$

First, we write the operator $3K_\infty + \tilde{\mu}$ in the form

$$3K_\infty + \tilde{\mu} = (3K_\infty + \mu_\infty)\left[1 - M_\varepsilon \ni_1^* \left(\tau_\mu^\varepsilon\right)\right], \tag{152}$$

where $M_\varepsilon = \mu_\infty \nu_\mu^\varepsilon (3K_\infty + \mu_\infty)^{-1}$.

In order to find an operator inverse to (152), let us introduce it in the form

$$(3K_\infty + \tilde{\mu})^{-1} = \frac{1}{3K_\infty + \mu_\infty}\left[1 + M_\sigma \ni_1^* (t_\sigma)\right], \tag{153}$$

where M_σ and t_σ are unknown constants for now.

Considering that $(3K_\infty + \tilde{\mu})(3K_\infty + \tilde{\mu})^{-1} = 1$, we obtain

$$\left[1 - M_\varepsilon \ni_1^* \left(\tau_\mu^\varepsilon\right)\right]\left[1 + M_\sigma \ni_1^* (t_\sigma)\right] = 1. \tag{154}$$

Opening the brackets in (154) and considering formula (18), we find

$$M_\sigma \ni_1^* (t_\sigma)\left(1 + M_\varepsilon \frac{t_\sigma}{\tau_\mu^\varepsilon - t_\sigma}\right) - M_\varepsilon \ni_1^* \left(\tau_\mu^\varepsilon\right)\left(1 + M_\sigma \frac{\tau_\mu^\varepsilon}{\tau_\mu^\varepsilon - t_\sigma}\right) = 0. \tag{155}$$

Equating to zero the expression in each bracket in (155) yields

$$\begin{cases} \tau_\mu^\varepsilon - t_\sigma + t_\sigma M_\varepsilon = 0, \\ \tau_\mu^\varepsilon - t_\sigma + \tau_\mu^\varepsilon M_\sigma = 0. \end{cases} \tag{156}$$

From the set of Eqs. (156) we find

$$M_\sigma = \frac{M_\varepsilon}{1 - M_\varepsilon}, \qquad t_\sigma = \frac{\tau_\mu^\varepsilon}{1 - M_\varepsilon}, \tag{157}$$

whence the known equality follows

$$\frac{\tau_\mu^\varepsilon}{t_\sigma} = \frac{M_\varepsilon}{M_\sigma}. \tag{158}$$

Finally, we can determine the operator $\widetilde{E}$ from the following relationship:

$$\widetilde{E} = E_\infty\left[1 - \nu_\mu^\varepsilon \ni_1^* \left(\tau_\mu^\varepsilon\right)\right]\left[1 + M_\sigma \ni_1^* (t_\sigma)\right]. \tag{159}$$

Opening the brackets in (159) and considering formula (18), we find

$$\widetilde{E} = E_\infty\left[1 - M_\sigma \frac{3K_\infty}{\mu_\infty} \ni_1^* (t_\sigma)\right]. \tag{160}$$

Substituting the operator $\ni_1^*(t_\sigma)$ in (160) by the operator $\ni_\gamma^*(t_\sigma^\gamma)$, we obtain the desired operator $\widetilde{E}$ which has the form

$$\widetilde{E} = E_\infty \left[1 - M_\sigma \frac{3K_\infty}{\mu_\infty} \ni_\gamma^* \left(t_\sigma^\gamma\right)\right], \tag{161}$$

in so doing

$$\left(\frac{t_\sigma}{\tau_\mu^\varepsilon}\right)^\gamma = \frac{M_\sigma}{M_\varepsilon}. \tag{162}$$

Thus, in the case under consideration, the operator $\widetilde{E} = E_\infty[1 - \nu_\varepsilon \ni_\gamma^*(\tau_\varepsilon^\gamma)]$ should be replaced by operator (161).

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