Dynamic response of a hereditarily elastic beam with Rabotnov's kernel impacted by an elastic rod

Yury A. Rossikhin[1], Marina V. Shitikova[1] and Ivan I. Popov[1,2]
[1]Research Center on Dynamics of Solids and Structures
Voronezh State University of Architecture and Civil Engineering
Voronezh 394006, Russian Federation
[2] National Taiwan University of Science and Technology, Taipei, R.O.C.
Email: yar@vgasu.vrn.ru mvs@vgasu.vrn.ru

Dedicated to the 100th Birthday of Russian Academician Yury N. Rabotnov

Abstract – The problem on low-velocity impact of a long thin elastic rod with a flat end upon an infinite Timoshenko-type beam, the viscoelastic features of which are exhibited only within the contact domain and are governed by the fractional derivative standard linear solid model, is formulated. The part of the beam being out of the contact region is considered to be elastic, and its behavior is described by a set of equations taking the rotary inertia and transverse shear deformation into account. At the moment of impact, shock waves are generated both in the impactor and target, the influence of which on the contact domain is considered via the theory of discontinuities. The contact zone moves like a rigid whole under the action of the contact force and transverse forces applied to the boundary of the contact region, which are obtained on the basis of one-term ray expansions. The contact force has been determined analytically via the Laplace transform technique.

Keywords–Impact response, hereditarily elastic Timoshenko-like beam, fractional derivative standard linear solid model, ray method, dynamic conditions of compatibility, Laplace transform.

I. Introduction

The problems connected with the analysis of the shock interaction of thin bodies (rods, beams, plates, and shells) with other bodies have widespread application in various fields of science and technology. The physical phenomena involved in the impact event include structural responses, contact effects and wave propagation. These problems are topical not only from the point of view of fundamental research in applied mechanics, but also with respect to their applications. Because these problems belong to the problems of dynamic contact interaction, their solution is connected with severe mathematical and calculation difficulties. To overcome this impediment, a rich variety of approaches and methods have been suggested, and the overview of current results in the field can be found in recent state-of-the-art articles [1]–[4].

In recent decades fractional calculus (integral and differential operators of noninteger order), which has a long history [5], has been the object of ever increasing interest in many branches of natural science, and of engineering interest as well. Thus, Rossikhin and Shitikova [3] have reviewed the application of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids, among them, the problems of dynamic contact interaction. Two approaches have been discussed for studying the impact response of fractionally damped systems subjected to falling impactors. The first one is based on the assumption that viscoelastic properties of the target manifest themselves only in the contact domain, while the other part of the target remains elastic one. This approach results in defining the contact force and the local penetration of target by an impactor from the set of linear fractional differential equations. The second approach is the immediate generalization of the Timoshenko approach utilizing the viscoelastic analog of Hertz's contact law by substituting elastic constants by the corresponding viscoelastic operators. This approach results in the nonlinear functional equation for determining the contact force or the impactor's relative displacement.

In the present paper, the analytical approach proposed in [2], [6] for the analysis of the dynamic response of the elastic isotropic Timoshenko beam subjected to the impact by elastic long rod has been extended to the problem of the dynamic response of a hereditarily elastic Timoshenko-like beam impacted by an elastic prismatic long rod of a rectangular cross-section. As this takes place, the impact response of thin isotropic beams is investigated under the assumption that the viscosity of the target exhibits only within the contact domain, while out of the contact region the beam remains to be elastic with a non-relaxed elastic modulus, in so doing viscous features are described by the fractional derivative standard linear solid model.

II. Problem Formulation

Let a long prismatic elastic rod of a rectangular cross-section with the dimensions $2\tau_{\rm im}$ and a move along the z-normal with the velocity V_0 towards an isotropic rectangular Timoshenko beam of infinite extent (this assumption is introduced due to the short duration of contact interaction in order to ignore reflected waves) with width a and thickness h, in so doing the normal z is erected at the middle of the beam.

The beam out of the contact zone is considered to be elastic, while within the contact domain its microstructure changes and it gains viscoelastic properties, which are described by the generalized fractional-derivative standard linear solid model. For the projectile with a flat end, such a scheme could be

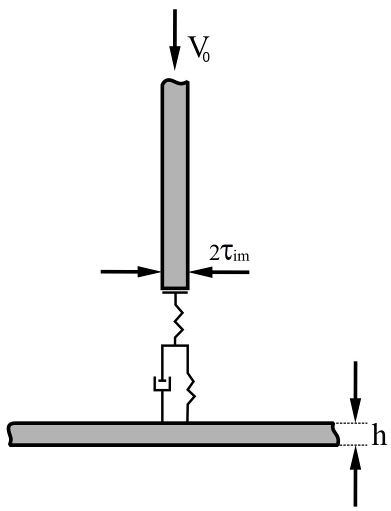

Fig. 1. Scheme of the shock interaction of a plain-end impactor with a target

realized if a viscoelastic buffer involving two springs and a viscous damper is embedded by its low end in the target (Figure 1).

Thus, the rod falls vertically upon the target. Impact occurs at $t = 0$ at the origin of the coordinate system x, y, z. At the moment of impact, shock waves (surfaces of strong discontinuity) are generated in the beam and in the rod, which then propagate along the projectile and the target with the velocities of the transient waves.

Further we will assume that during the process of impact the transverse forces and transverse shear deformations dominate in the stress-strain state of the beam within the vicinity of the contact zone. Besides, the elastic rod and the beam are considered to be somewhat extended, so that the waves reflected from rod's free edge and beam's boundary have had no time to return to the contact region to terminate the collision.

III. Governing Equations

The dynamic behavior of an elastic homogeneous prismatic beam with due account for the rotary inertia and transverse shear deformations is described by the following set of equations [2], [6]:

$$\frac{\partial Q_r}{\partial z} = \rho A \dot{W}, \tag{1}$$

$$\frac{\partial M}{\partial z} - Q = -\rho I \dot{\beta}, \tag{2}$$

$$\dot{Q}_r = K\mu_\infty A\left(\partial W/\partial z - \beta\right), \tag{3}$$

$$\dot{M} = -E_\infty I \partial\beta/\partial z, \tag{4}$$

where M is the bending moment, Q is the shear force, $W = \dot{w}$ is the transverse displacement velocity of a beam central axis (velocity of deflection), β is the angular velocity of a cross-section about the z-axis which is perpendicular to the plane of flexure $y - z$ (the axes z and y are directed along the beam axis and vertically down, respectively), E_∞ and μ_∞ are the nonrelaxed magnitudes of the elastic and shear moduli corresponding to the elastic beam, respectively, ρ is the density, K is the shear coefficient, A and I are the cross-sectional area and the moment of inertial with respect to the z-axis, respectively, and an overdot denotes the time derivative.

To equations (1) to (4), one should add equations describing the dynamic behavior of the elastic rod (impactor)

$$\frac{\partial \sigma}{\partial z} = \rho_{\rm im} \dot{v}, \tag{5}$$

$$\dot{\sigma} = E_{\rm im}\, \frac{\partial v}{\partial z}, \tag{6}$$

where σ is the stress, v is the velocity, $\rho_{\rm im}$ and $E_{\rm im}$ are the density and Young's modulus of impactor's material, respectively, as well as the equation of motion of the contact domain of the length $2\tau_{\rm im}$ (Figure 1)

$$2\tau_{\rm im} A\rho \ddot{w} = 2Q|_{z=\tau_{\rm im}} + F_{\rm cont}, \tag{7}$$

and the equation for the contact force which could be written as the fractional derivative standard linear solid constitutive relationship

$$F_{\rm cont} + \tau_\varepsilon^\gamma D^\gamma F_{\rm cont} = E_0\left[(\alpha - w) + \tau_\sigma^\gamma D^\gamma (\alpha - w)\right], \tag{8}$$

wherein α and w are the displacements of the upper and lower ends of the buffer, respectively, in so doing the displacement w is equal to the displacement of the beam in the place of contact (Figure 1), γ $(0 < \gamma \le 1)$ is the fractional parameter, τ_ε and τ_σ are the relaxation and retardation (creep) times, respectively, in so doing

$$\tau_\varepsilon^\gamma \tau_\sigma^{-\gamma} = E_0 E_\infty^{-1}, \tag{9}$$

E_∞ and E_0 are the non-relaxed (instantaneous modulus of elasticity, or the glassy modulus) and relaxed elastic (prolonged modulus of elasticity, or the rubbery modulus) moduli, respectively,

$$D^\gamma F_{\rm cont} = \frac{\rm d}{{\rm d}t} \int\limits_0^t \frac{F_{\rm cont}(t - t')}{\Gamma(1-\gamma) t'^\gamma}\, {\rm d}t' \tag{10}$$

is the Riemann-Liouville fractional derivative, and $\Gamma(1-\gamma)$ is the Gamma-function.

The above equations are subjected to the initial conditions

$$\alpha|_{t=0} = w|_{t=0} = \dot{w}|_{t=0} = 0, \quad \dot{\alpha}|_{t=0} = V_0, \tag{11}$$

as well as the boundary condition

$$\partial W/\partial z|_{z=\pm\tau_{\rm im}} = 0. \tag{12}$$

IV. Methods of Solution

To find the solution of the stated problem, two methods are used, namely: the ray method and Laplace transform technique. The ray method is applied for constructing an approximate solution within the elastic part of the beam from the surface of strong discontinuity upto the boundary of the contact region, as well as for finding the exact solution within the disturbed domain of the elastic rod. Within the contact domain, the Laplace transformation method is utilized to determine the contact force.

A. A Ray Method for the Elastic Part of the Beam

To find the solution outward the contact region, i.e., for the elastic part of the target, we shall interpret a shock wave in the beam (surface of strong discontinuity) as a layer of the thickness δ, within which the desired function Z changes from the magnitude Z^- to the magnitude Z^+ but remaining a continuous function [2]. Then integrating equations from (1) to (4) over the layer's thickness from $-\delta/2$ to $\delta/2$, with δ tending to zero, and considering that inside the layer the condition of compatibility [7] is fulfilled in the form of

$$\dot{Z} = -\, G\, \frac{\partial Z}{\partial r} + \frac{\delta Z}{\delta t}, \tag{13}$$

where G is the normal velocity of the wave surface, and $\delta/\delta t$ is the δ-derivative with respect to time [8], we find the dynamic conditions of compatibility

$$[Q] = -\,\rho A G[W], \qquad -\,G[Q] = K\mu_\infty A[W], \tag{14}$$

$$[M] = -\,\rho I\; G[\beta], \qquad -\,G[M] = E_\infty I[\beta], \tag{15}$$

where $[Z] = Z^+ - Z^-$.

Eliminating the values $[Q]$ and $[M]$ from equations (14) and (15), respectively, we define the velocities of the quasi-transverse $G_\infty^{(2)}$ and quasi-longitudinal $G_\infty^{(1)}$ waves as follows

$$G_\infty^{(2)} = \left(\frac{K\mu_\infty}{\rho}\right)^{1/2}, \qquad G_\infty^{(1)} = \left(\frac{E_\infty}{\rho}\right)^{1/2}. \tag{16}$$

If the contact spot is considered to be a rigid body, then the values $Q \approx [Q]$ and $W \approx [W]$, which are connected by the relationship

$$Q = -\,\rho A G_\infty^{(2)} W, \tag{17}$$

are the dominating values in the vicinity of the contact spot and on its boundary [2].

B. Ray Method for the Elastic Rod

At the moment of impact of a projectile (rod) against a target (beam), the shock waves are generated not only in the beam but in the rod (a longitudinal shock wave) as well, which propagates along the rod with the velocity $G_{\rm im}$.

Using the same reasoning for determining the dynamic conditions of compatibility as we have adopted above for the elastic beam, we find

$$[\sigma] = -\,\rho_{\rm im} G_{\rm im}[v], \qquad -\,G_{\rm im}[\sigma] = E_{\rm im}[v], \tag{18}$$

whence it follows that

$$G_{\rm im} = \sqrt{\frac{E_{\rm im}}{\rho_{\rm im}}}. \tag{19}$$

Behind the front of this wave (a surface of the strong discontinuity), the relationships for the stress σ^- and velocity v^- could be obtained using the ray series [2]

$$\sigma^- = -\sum_{k=0}^{\infty} \frac{1}{k!}\; \left[\sigma_{,(k)}\right] \left(t - \frac{z}{G_{\rm im}}\right)^k, \tag{20}$$

$$v^- = V_0 - \sum_{k=0}^{\infty} \frac{1}{k!}\; \left[v_{,(k)}\right] \left(t - \frac{z}{G_{\rm im}}\right)^k, \tag{21}$$

where $\sigma_{,(k)} = \partial^k/\partial t^k$ and $v_{,(k)} = \partial^k/\partial t^k$.

Considering that the discontinuities in the elastic rod remain constant during the process of the wave propagation, and using the condition of compatibility

$$G_{\rm im} \left[\frac{\partial Z_{,(k-1)}}{\partial z}\right] = -[Z_{,(k)}],$$

which is obtained from equation (13) by substitution of the function Z with $Z_{,(k)} = \partial^k Z/\partial t^k$, we have

$$\left[\frac{\partial \sigma_{,(k-1)}}{\partial z}\right] = -G_{\rm im}^{-1}[\sigma_{,(k)}]. \tag{22}$$

With due account for (22), the equation of motion on the wave surface is written in the form

$$[\sigma_{,(k)}] = -\rho_{\rm im} G_{\rm im}[v_{,(k)}]. \tag{23}$$

Substituting (23) in (20) yields

$$\sigma^- = \rho_{\rm im} G_{\rm im} \sum_{k=0}^{\infty} \frac{1}{k!}\; \left[v_{,(k)}\right] \left(t - \frac{z}{G_{\rm im}}\right)^k. \tag{24}$$

Comparing relationships (24) and (21), we obtain

$$\sigma^- = \rho_{\rm im} G_{\rm im}(V_0 - v^-). \tag{25}$$

At $z = 0$, expression (25) takes the form

$$\sigma_{\rm cont} = \rho_{\rm im} G_{\rm im}(V_0 - W - \dot{\alpha}), \tag{26}$$

where $\sigma_{\rm cont} = \sigma^-|_{z=0}$ is the contact stress.

Using formula (26), it is possible to find the contact force

$$F_{\rm cont} = b(V_0 - \dot{w} - \dot{\alpha}), \tag{27}$$

where $b = 2\tau_{\rm im} a \rho_{\rm im} G_{\rm im}$.

C. Determination of the Contact Force by the Laplace Transform Technique

The contact force can be determined not only by formula (27) but according to the following equation [9] as well:

$$F_{\rm cont} = E_\infty(\alpha - w) - \triangle E \int_0^t \ni_\gamma \left(-\frac{t-t'}{\tau_\varepsilon}\right) [\alpha(t') - w(t')] {\rm d}t', \tag{28}$$

where $\triangle E = E_\infty - E_0$ is the defect of the modulus, i.e., the value characterizing the decrease in the elastic modulus from its nonrelaxed value to its relaxed value, and

$$\ni_\gamma \left(-\frac{t}{\tau_\varepsilon}\right) = \frac{t^{\gamma-1}}{\tau_\varepsilon^\gamma} \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau_\varepsilon)^{\gamma n}}{\Gamma[\gamma(n+1)]} \tag{29}$$

is the fractional exponential function suggested by Rabotnov [10].

Really, we could rewrite equation (8) in the form

$$F_{\rm cont} = E_0\, \frac{1 + \tau_\sigma^\gamma D^\gamma}{1 + \tau_\varepsilon^\gamma D^\gamma}(\alpha - w), \tag{30}$$

or with due account for formula (9) in the form

$$F_{\rm cont} = E_\infty \, \frac{E_\infty^{-1} E_0 + \tau_\varepsilon^\gamma D^\gamma}{1+\tau_\varepsilon^\gamma D^\gamma} (\alpha - w). \tag{31}$$

Adding and subtracting the unit in the numerator of equation (31) yields

$$F_{\rm cont} = E_\infty (\alpha - w) - \triangle E \ni_\gamma^* (\tau_\varepsilon^\gamma)\,(\alpha - w), \tag{32}$$

where

$$\ni_\gamma^* (\tau_\varepsilon^\gamma) = \frac{1}{1+\tau_\varepsilon^\gamma D^\gamma}$$

is the dimensionless Rabotnov operator [11].

Considering that $D^\gamma I^\gamma = 1$, we could represent the operator $\ni_\gamma^* (\tau_\varepsilon^\gamma)$ as

$$\ni_\gamma^* (\tau_\varepsilon^\gamma) = \frac{I^\gamma \tau_\varepsilon^{-\gamma}}{1-\left(-I^\gamma \tau_\varepsilon^{-\gamma}\right)}, \tag{33}$$

where

$$I^\gamma x(t) = \int\limits_0^t \frac{(t-t')^{\gamma-1}}{\Gamma(\gamma)} \, x(t'){\rm d}t' \tag{34}$$

is the fractional integral.

If we suppose that the right part of formula (33) is the sum of an infinite decreasing geometrical progression, the denominator of which is equal to $d = -I^\gamma \tau_\varepsilon^{-\gamma}$, then $\ni_\gamma^* (\tau_\varepsilon^\gamma)$ could be represented as

$$\ni_\gamma^* (\tau_\varepsilon^\gamma) = \sum_{n=0}^{\infty} (-1)^n \tau_\varepsilon^{-\gamma(n+1)} I^{\gamma(n+1)}, \tag{35}$$

or with due account for equation (34), we find

$$\ni_\gamma^* (\tau_\varepsilon^\gamma)\, x(t) = \int\limits_0^t \ni_\gamma \left(-\frac{t'}{\tau_\varepsilon}\right) x(t-t'){\rm d}t'. \tag{36}$$

If we change the subtrahend in equation (32) by formula (36) with $x(t) = \alpha(t) - w(t)$, then we are led to relationship (28).

Equations (27), (28) and (7) rewritten with due account for formula (17)

$$M\ddot{w} + MB\dot{w} = F_{\rm cont}, \tag{37}$$

where $B = \tau_{\rm im}^{-1} G_\infty^{(2)}$ and $M = 2\tau_{\rm im}\rho A$ is the mass of the contact region, provide a closed set of three equations in terms of three unknowns: $F_{\rm cont}$, w, and α.

Now applying Laplace transformation to equations (37), (30), and (27), we have

$$Mp\bar{w}(p+B) = \bar{F}_{\rm cont}, \tag{38}$$

$$\bar{F}_{\rm cont} = E_0 \, \frac{1+(p\tau_\sigma)^\gamma}{1+(p\tau_\varepsilon)^\gamma} \left(\bar{\alpha} - \bar{w}\right), \tag{39}$$

$$\bar{F}_{\rm cont} = b\left(\frac{V_0}{p} - p\bar{\alpha} - p\bar{w}\right), \tag{40}$$

where a bar over a value denotes the Laplace transform the given value, and p is the Laplace variable.

Eliminating $\bar{F}_{\rm cont}$ from equations (38) and (40), we find

$$\bar{\alpha}(p) = \frac{V_0}{p^2} - \left[\frac{M}{b}(p+B)+1\right]\bar{w}. \tag{41}$$

Now eliminating $\bar{F}_{\rm cont}$ from equations (38) and (39) and considering (41), we obtain

$$\bar{w}(p) = \frac{V_0 \Omega_\infty^2 (\tau_\sigma^{-\gamma} + p^\gamma)}{p^2 f_\gamma(p)}, \tag{42}$$

where $\Omega_\infty^2 = E_\infty M^{-1}$, and

$$\begin{aligned} f_\gamma(p) &= p^{2+\gamma} + \tau_\varepsilon^{-\gamma} p^2 + (B + E_\infty b^{-1}) p^{1+\gamma} \\ &+ (B + E_0 b^{-1}) \tau_\varepsilon^{-\gamma} p + E_\infty (Bb^{-1} + M^{-1}) p^\gamma \\ &+ E_0 (Bb^{-1} + M^{-1}) \tau_\varepsilon^{-\gamma}. \end{aligned} \tag{43}$$

Substituting formulas (41) and (42) in (40) yields

$$\bar{F}_{\rm cont}(p) = \frac{V_0 E_\infty (p+B)(\tau_\sigma^{-\gamma} + p^\gamma)}{p f_\gamma(p)}. \tag{44}$$

Besides, it is possible to find the value $\bar{\alpha}(p)$, if we exclude the value $\bar{w}(p)$ defined by (42) from equation (41). As a result, we obtain

$$\bar{\alpha}(p) = \frac{V_0}{p^2}\left\{1 - \left[\frac{M}{b}(p+B)+1\right] \frac{\Omega_\infty^2 (\tau_\sigma^{-\gamma} + p^\gamma)}{f_\gamma(p)}\right\}. \tag{45}$$

Now we will carry out the inverse transformation of formula (44). For this purpose, first we will investigate the roots of the characteristic equation

$$f_\gamma(p) = 0. \tag{46}$$

Let us multiply equation (46) by τ_ε^γ, represent p in the geometrical form

$$p = re^{{\rm i}\psi} \tag{47}$$

and introduce a new variable $x = (r\tau_\varepsilon)^\gamma$. As a result, equation (46) could be rewritten in the form

$$\begin{aligned} & r^2 \left[x e^{i(2+\gamma)\psi} + e^{2i\psi}\right] \\ + \; & r\left[(B + E_\infty b^{-1}) x e^{i(1+\gamma)\psi} + (B + E_0 b^{-1}) e^{i\psi}\right] \\ + \; & \left(Bb^{-1} + 2M^{-1}\right)\left(E_\infty x e^{i\gamma\psi} + E_0\right) = 0. \end{aligned} \tag{48}$$

Separating the real and imaginary parts in (48), we have

$$r^2 a_1 + r a_2 + a_3 = 0, \tag{49}$$

$$r^2 b_1 + r b_2 + b_3 = 0, \tag{50}$$

where

$$\begin{aligned} a_1 &= \cos 2\psi + x\cos(2+\gamma)\psi, \\ b_1 &= \sin 2\psi + x\sin(2+\gamma)\psi, \\ a_2 &= (B + E_0 b^{-1})[\cos\psi + x(B + E_\infty b^{-1})\cos(1+\gamma)\psi], \\ b_2 &= (B + E_0 b^{-1})[\sin\psi + x(B + E_\infty b^{-1})\sin(1+\gamma)\psi], \\ a_3 &= \left(Bb^{-1} + 2M^{-1}\right)(E_0 + xE_\infty \cos\gamma\psi), \\ b_3 &= \left(Bb^{-1} + 2M^{-1}\right) xE_\infty \sin\gamma\psi. \end{aligned}$$

First we fix the angle $\frac{\pi}{2} \le \psi \le \pi$ in equations (49) and (50), and then eliminate r^2. As result, we obtain

$$r = \frac{a_1 b_3 - a_3 b_1}{a_2 b_1 - a_1 b_2}. \tag{51}$$

Substituting (51) in (49) yields

$$\begin{aligned}(a_3 b_1 - a_1 b_3)^2 a_1 \quad &- \quad (a_2 b_1 - a_1 b_2)(a_3 b_1 - a_1 b_3) a_2 \\ &+ \quad (a_2 b_1 - a_1 b_2)^2 a_3 = 0. \end{aligned}\tag{52}$$

From equation (52) at each fixed angle ψ from the segment $\frac{\pi}{2} \le \psi \le \pi$, we could find the values x_i $(i = 1, 2, ...)$, and then we substitute the chosen ψ with the found magnitude of x_i in equation (51), what allows us to find the corresponding module r_i $(i = 1, 2, ...)$. Knowing the values of x_i and r_i, it is possible to determine $(\tau_\varepsilon^\gamma)_i = x_i r_i^{-\gamma}$. The set of values involving the angle ψ, radii r_i, and parameters $(\tau_\varepsilon^\gamma)_i$ completely defines the roots of the characteristic equation (46).

In order to clarify the number of characteristic equation roots, we consider their asymptotic behavior.

1) The case $\tau_\varepsilon^\gamma \to 0$: Suppose that $\tau_\varepsilon^\gamma \to 0$ $(\tau_\varepsilon^{-\gamma} \to \infty)$. In this case, the characteristic equation (46) takes the form

$$f_{\gamma 0}(p_0) = p_0^2 + (B + E_0 b^{-1}) p_0 + E_0 (B b^{-1} + M^{-1}) = 0, \tag{53}$$

whence it follows that

$$\begin{aligned} p_{0i} = (p_0)_{1,2} = \quad &- \quad \frac{1}{2}(B + E_0 b^{-1}) \\ &\pm \quad \frac{1}{2}\sqrt{(B - E_0 b^{-1})^2 - 8 E_0 M^{-1}}. \end{aligned}\tag{54}$$

2) The case $\tau_\varepsilon^\gamma = \varepsilon$: Now we suppose that the relaxation time of the system is a small value, i.e. $\tau_\varepsilon^\gamma = \varepsilon$, where ε is a small value. We will seek the solution of the characteristic equation (46) in the form:

$$p_i = p_{0i} + \varepsilon \chi_i, \tag{55}$$

where χ_i is yet unknown function.

Substituting (55) in (46) and ignoring the values of the order higher than ε, we find

$$\chi_i = -\frac{f_{\gamma\infty}(p_{0i})}{f'_{\gamma 0}(p_{0i})}, \tag{56}$$

where $f'_\gamma(p)$ denotes the derivative of the function $f_\gamma(p)$ with respect to p,

$$f'_\gamma(p_{0i}) = 2p_{0i} + B + E_0 b^{-1},$$

$$f_{\gamma\infty} = p_{0i}^{2+\gamma} + (B + E_\infty b^{-1}) p_{0i}^{1+\gamma} + E_\infty (B b^{-1} + 2M^{-1}) p_{0i}^{\gamma}.$$

3) The case $\tau_\varepsilon^\gamma \to \infty$: Suppose that $\tau_\varepsilon^\gamma \to \infty$ $(\tau_\varepsilon^{-\gamma} \to 0)$. In this case, the characteristic equation (46) takes the form

$$\begin{aligned} f_{\gamma\infty}(p_\infty) \quad &= \quad p_\infty^{2+\gamma} + (B + E_\infty b^{-1}) p_\infty^{1+\gamma} \\ &+ \quad E_\infty (B b^{-1} + 2M^{-1}) p_\infty^{\gamma} = 0. \end{aligned}\tag{57}$$

From equation (57) we find

$$\begin{aligned} p_{\infty i} = (p_\infty)_{1,2} = &- \frac{1}{2}(B + E_\infty b^{-1}) \\ &\pm \frac{1}{2}\sqrt{(B - E_\infty b^{-1})^2 - 8 E_\infty M^{-1}}. \end{aligned}\tag{58}$$

4) The case $\tau_\varepsilon^{-\gamma} = \varepsilon$: Now we suppose that the relaxation time of the system is a large value, i.e. $\tau_\varepsilon^{-\gamma} = \varepsilon$, where ε is a small value. We will seek the solution of the characteristic equation (46) in the form:

$$p_i = p_{\infty i} + \varepsilon \eta_i. \tag{59}$$

Substituting (59) in equation (46) and ignoring the values of the order higher than ε, we find

$$\eta_i = -\frac{f_{\gamma 0}(p_{\infty i})}{f'_{\gamma\infty}(p_{\infty i})}, \tag{60}$$

where

$$f_{\gamma 0}(p_{\infty i}) = p_{\infty i}^2 + (B + E_0 b^{-1}) p_{\infty i} + E_0 (B b^{-1} + 2M^{-1}),$$

$$\begin{aligned} f'_{\gamma\infty}(p_{\infty i}) = (2+\gamma) p_{\infty i}^{1+\gamma} + (B + E_\infty b^{-1})(1+\gamma) p_{\infty i}^{\gamma} \\ + (B b^{-1} + 2M^{-1}) \gamma p_{\infty i}^{\gamma - 1}. \end{aligned}$$

On the ground of the above asymptotic formulas, it could be assumed that the characteristic equation (46) possesses two complex conjugate roots, which we will represent in the following form:

$$p_{1,2} = r e^{\pm \mathrm{i}\psi} = -\alpha \pm \mathrm{i}\omega. \tag{61}$$

Further it is convenient to rewrite $\bar{F}_{\mathrm{cont}}(p)$ defined by (44) in the form

$$\bar{F}_{\mathrm{cont}}(p) = \frac{1}{p}\, \bar{F}_0(p), \tag{62}$$

where

$$\bar{F}_0(p) = V_0\, \frac{g_\gamma(p)}{f_\gamma(p)}, \tag{63}$$

$$g_\gamma(p) = E_\infty p^{1+\gamma} + E_0 \tau_\varepsilon^{-\gamma} p + E_\infty B p^{\gamma} + E_0 \tau_\varepsilon^{-\gamma} B.$$

The function $F_0(t)$ in the time domain is governed by the Mellin-Fourier inversion formula

$$F_0(t) = \frac{1}{2\pi \mathrm{i}} \int_{c - \mathrm{i}\infty}^{c + \mathrm{i}\infty} \bar{F}_0(p) e^{pt} \mathrm{d}p. \tag{64}$$

To calculate the integral (64), it is necessary to define all singular points of the complex function $\bar{F}_{\mathrm{cont}}(p)$. This multi-valued function possesses the branch points at $p = 0$ and $p = \infty$ and the simple poles at the same magnitudes of $p = p_k$ which vanish to zero the denominator in equation (63), i.e. they are the roots of the characteristic equation (46).

The inversion theorem is applicable to multi-valued functions possessing branch points only on the first sheet of the Riemann surface, i.e. when $0 < |\arg p| < \pi$. Thus a closed contour of integration should be chosen in the form presented in Figure 2. Considering Jordan lemma and applying the main theorem of the theory of residues, we rewrite the integral (64) in the following form:

$$\begin{aligned} F_0(t) \quad &= \quad \frac{1}{2\pi \mathrm{i}} \int_0^\infty \left[\bar{F}_0(s e^{-\mathrm{i}\pi}) - \bar{F}_0(s e^{\mathrm{i}\pi}) \right] e^{-st} \mathrm{d}s \\ &+ \quad \sum_k \mathrm{res} \left[\bar{F}_0(p_k) e^{p_k t} \right], \end{aligned}\tag{65}$$

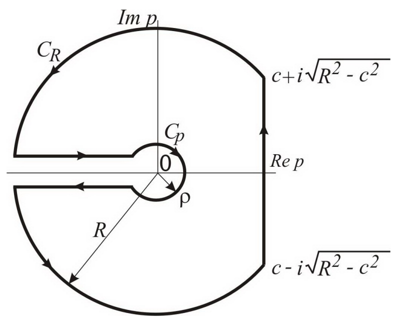

Fig. 2. Closed contour of integration

where the summation is carried out over all isolated singular points (poles).

Knowing the function $F_0(t)$, it is possible to determine the contact force $F_{\rm cont}(t)$ via the following formula:

$$F_{\rm cont}(t) = \int_0^t F_0(t')\mathrm{d}t'. \tag{66}$$

Since the roots of the characteristic equation (46) are complex conjugate ones and are defined by formula (61), then equation (65) is reduced to

$$F_0(t) = A_0(t) + A\exp(-\alpha t)\cos(\omega t + \varphi), \tag{67}$$

where

$$A_0(t) = \int_0^\infty B(s)e^{-st}\mathrm{d}s,$$

$$B(s) = \frac{(s-B)\left[Y\mathrm{Re}f_\gamma(se^{\mathrm{i}\pi}) - X\mathrm{Im}f_\gamma(se^{\mathrm{i}\pi})\right]V_0\pi^{-1}}{\left[\mathrm{Re}f_\gamma(se^{\mathrm{i}\pi})\right]^2 + \left[\mathrm{Im}f_\gamma(se^{\mathrm{i}\pi})\right]^2},$$

$$A_j = \frac{2V_0\sqrt{\left[N_1(p_j)\right]^2 + \left[N_2(p_j)\right]^2}}{\left[\mathrm{Re}f'_\gamma(p_j)\right]^2 + \left[\mathrm{Im}f'_\gamma(p_j)\right]^2}, \quad A_1 = A_2 = A,$$

$$\tan\varphi_j = \frac{\mathrm{Re}f'_\gamma(p_j)\mathrm{Re}\,g_\gamma(p_j) + \mathrm{Im}f'_\gamma(p_j)\mathrm{Im}\,g_\gamma(p_j)}{\mathrm{Im}f'_\gamma(p_j)\mathrm{Re}\,g_\gamma(p_j) - \mathrm{Re}f'_\gamma(p_j)\mathrm{Im}\,g_\gamma(p_j)},$$
$$\tan\varphi_2 = -\tan\varphi_2 = \tan\varphi,$$

$$\begin{aligned}
&\mathrm{Re}f_\gamma(se^{\mathrm{i}\pi}) = \tau_\varepsilon^{-\gamma}\left\{s^2\left[(s\tau_\varepsilon)^\gamma\cos(2+\gamma)\pi + 1\right]\right.\\
&+s\left[(s\tau_\varepsilon)^\gamma(B + E_\infty b^{-1})\cos(1+\gamma)\pi - (B + E_0b^{-1})\right]\\
&\left.+\left(Bb^{-1} + 2M^{-1}\right)\left[E_\infty(s\tau_\varepsilon)^\gamma\cos\gamma\pi + E_0\right]\right\}\\
&= \mathrm{Re}f_\gamma(se^{-\mathrm{i}\pi}),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Im}f_\gamma(se^{\mathrm{i}\pi}) = \tau_\varepsilon^{-\gamma}\left[s^2(s\tau_\varepsilon)^\gamma\sin(2+\gamma)\pi\right.\\
&+s(s\tau_\varepsilon)^\gamma(B + E_\infty b^{-1})\sin(1+\gamma)\pi\\
&\left.+\left(Bb^{-1} + 2M^{-1}\right)E_\infty(s\tau_\varepsilon)^\gamma\sin\gamma\pi\right] = -\mathrm{Im}f_\gamma(se^{-\mathrm{i}\pi}),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Re}f'_\gamma(p_1) = (2+\gamma)r^{1+\gamma}\cos(1+\gamma)\psi + 2r\tau_\varepsilon^{-\gamma}\cos\psi\\
&+(1+\gamma)(B + E_\infty b^{-1})r\cos\gamma\psi\\
&+\gamma E_\infty(Bb^{-1} + 2M^{-1})r^{\gamma-1}\cos(\gamma-1)\psi\\
&+(B + E_0b^{-1})\tau_\varepsilon^{-\gamma} = \mathrm{Re}f'_\gamma(p_2),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Im}f'_\gamma(p_1) = (2+\gamma)r^{1+\gamma}\sin(1+\gamma)\psi + 2r\tau_\varepsilon^{-\gamma}\sin\psi\\
&+(1+\gamma)(B + E_\infty b^{-1})r\sin\gamma\psi\\
&+\gamma E_\infty(Bb^{-1} + 2M^{-1})r^{\gamma-1}\sin(\gamma-1)\psi = -\mathrm{Im}f'_\gamma(p_2),
\end{aligned}$$

$$X = E_0\tau_\varepsilon^{-\gamma} + E_\infty s^\gamma\cos\pi\gamma, \qquad Y = E_\infty s^\gamma\sin\pi\gamma,$$

$$N_1(p_j) = \mathrm{Re}f'_\gamma(p_j)\mathrm{Re}\,g_\gamma(p_j) + \mathrm{Im}f'_\gamma(p_j)\mathrm{Im}\,g_\gamma(p_j),$$

$$N_1(p_1) = N_1(p_2),$$

$$N_2(p_j) = \mathrm{Im}f'_\gamma(p_j)\mathrm{Re}\,g_\gamma(p_j) - \mathrm{Re}f'_\gamma(p_j)\mathrm{Im}\,g_\gamma(p_j),$$

$$N_2(p_1) = -N_2(p_2),$$

$$\begin{aligned}
&\mathrm{Re}\,g_\gamma(p_1) = E_\infty r^{1+\gamma}\cos(1+\gamma)\psi + E_0\tau_\varepsilon^{-\gamma}r\cos\psi\\
&+BE_\infty r^\gamma\cos\gamma\psi + BE_0\tau_\varepsilon^{-\gamma} = \mathrm{Re}\,g_\gamma(p_2),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Im}\,g_\gamma(p_1) = E_\infty r^{1+\gamma}\sin(1+\gamma)\psi + E_0\tau_\varepsilon^{-\gamma}r\sin\psi\\
&+BE_\infty r^\gamma\sin\gamma\psi = -\mathrm{Im}\,g_\gamma(p_2).
\end{aligned}$$

The first term in equation (67) defines the drift of the equilibrium position, while the second term governs damping vibrations around the drifting equilibrium position.

According to equation (66), for determining the function $F_{\rm cont}(t)$, it is a need to integrate the function (67) over t from o to t. As a result we obtain

$$\begin{aligned}
F_{\rm cont}(t) = &\int_0^\infty B(s)\left(1 - e^{-st}\right)\mathrm{d}s\\
&+\frac{A}{\alpha^2+\omega^2}\Big\{\alpha\sin\varphi + \omega\cos\varphi\\
&-e^{-\alpha t}\left[\alpha\sin(\omega t+\varphi) + \omega\cos(\omega t+\varphi)\right]\Big\}.
\end{aligned} \tag{68}$$

V. Conclusion

The impact of a long thin cylindrical plain-ended elastic rod upon an infinite isotropic rectangular prismatic beam is investigated for the case when the viscoelastic features of the beam represent themselves only in the place of contact as a result of changes of target's microstructure during the process of contact interaction and are governed by the standard linear solid model with fractional derivatives. Out of the contact domain the target remains elastic with the non-relaxed magnitude of the elastic modulus. Due to the short duration of contact interaction, the reflected waves are not taken into account. In other words, it is assumed that the impactor will bounce from the target before the reflected waves have a time to reach the place of contact. The problem of determining the contact force is a quasi-linear one, and the Laplace transform technique has been used for its analytical solution.

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